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MATHEMATICS

On the Estimation of $(\bar{N}, p_n)$-Means of Orthogonal Series

by

J. MEDER

Presented by W. ORLICZ on December 17, 1960

1. A series

$$u_0 + u_1 + \ldots + u_n + \ldots$$

with the n-th partial sums s_n, is said to be summable $(\bar{N}, p_n)$ to the value s, if $p_n \geqslant 0$, $p_0 > 0$, $\sum^{\infty} p_n = \infty$ (so that $P_n = p_0 + p_1 + \ldots + p_n \to \infty$), and

$$T_n = \frac{p_0 s_0 + p_1 s_1 + \ldots + p_n s_n}{p_0 + p_1 + \ldots + p_n} \to s, \text{ when } n \to \infty .$$

The expressions T_n will be called the $(\bar{N}, p_n)$-means of the series $\sum u_n$ ([1], p. 57).

In the present paper we are concerned with the estimation of $(\bar{N}, p_n > 0)$-means of orthogonal series.

Let ON $\{\varphi_n(x)\}$ denote an orthonormal system defined in interval $\langle 0, 1 \rangle$.

Further, let the orthogonal series corresponding to the system ON $\{\varphi_n(x)\}$ be

$$\sum_{n=0}^{\infty} a_n \varphi_n(x) \tag{1.1}$$

with real coefficients a_n satisfying the condition $\{a_n\} \in l^2$, i.e.

$$\sum_{n=0}^{\infty} a_n^2 < \infty . \tag{1.2}$$

Given a sequence $\{a_n\} \in l^2$, we denote by $\{\lambda_n\}$ a sequence of positive numbers increasing to infinity, and such, that

$$\sum_{n=0}^{\infty} a_n^2 \lambda_n^2 < \infty , \tag{1.3}$$

(such a sequence exists, see e.g. [4], p. 101). Denoting

$$(1.4)\qquad \begin{cases} s_n(x) = a_0\varphi_0(x) + \ldots + a_n\varphi_n(x), \\ s_n^*(x) = a_0\lambda_0\varphi_0(x) + \ldots + a_n\lambda_n\varphi_n(x), \\ T_n(x) = \dfrac{1}{P_n}(p_0 s_0(x) + \ldots + p_n s_n(x)), \\ T_n^*(x) = \dfrac{1}{P_n}(p_0 s_0^*(x) + \ldots + p_n s_n^*(x)), \end{cases}$$

we formulate the following

LEMMA 1. *Let* $\{w(n)\}$ *be a non-decreasing sequence of positive numbers tending to infinity,* $\{a_n\} \in l^2$, *and*

$$T_n^*(x) \doteq O[w(n)], \text{ when } n \to \infty \text{ *}),$$

then

$$T_n(x) \doteq o[w(n)], \text{ when } n \to \infty.$$

This Lemma has been proved by the author in a recently published paper ([3], Lemma 3, p. 8).

2. THEOREM 1.**) *If* $\{a_n\} \in l^2$,

then

$$T_n(x) \doteq o(\log\log P_n), \text{ when } n \to \infty.$$

Proof. Let $\{\mu_n\}$ denote an arbitrary sequence of positive numbers increasing to infinity.

Let us denote

$$\bar{s}_n^*(x) = \sum_{\nu=0}^{n} \frac{a_\nu\lambda_\nu\varphi_\nu(x)}{\mu_\nu}; \quad T(x) = \frac{1}{P_n}\sum_{k=0}^{n} p_k \bar{s}_k^*(x).$$

Applying to the last expression the Abel transformation and the notations (1.4) we can write

$$\bar{T}_n^*(x) = \frac{1}{P_n}\sum_{k=0}^{n} p_k \sum_{\nu=0}^{k} \frac{a_\nu\lambda_\nu\varphi_\nu(x)}{\mu_\nu} = \frac{1}{P_n}\sum_{k=0}^{n} \frac{p_k s_k^*(x)}{\mu_k} +$$

$$+ \frac{1}{P_n}\sum_{k=0}^{n} p_k \sum_{\nu=0}^{k-1} s_\nu^*(x)\,\Delta\frac{1}{\mu_\nu} = \frac{1}{\mu_n}\frac{1}{P_n}\sum_{k=0}^{n} p_k s_k^*(x) +$$

$$+ \frac{1}{P_n}\sum_{k=0}^{n-1}\sum_{\nu=0}^{k} p_\nu s_\nu^*(x)\,\Delta\frac{1}{\mu_k} + \frac{1}{P_n}\sum_{k=0}^{n}\sum_{\nu=0}^{k-1} s_\nu^*(x)\,\Delta\frac{1}{\mu_\nu}.$$

*) If the sequence $\{f_n(x)/g_n(x)\}$ in $<0,1>$ is bounded or convergent to zero for $n \to \infty$ almost everywhere, then we shall write

$$f_n(x) \doteq O[g_n(x)] \text{ or } f_n(x) \doteq o[g_n(x)],$$

respectively

**) This theorem and the next are a generalization of similar results obtained recently by the author for the first logarithmic means ([3], Th. 5 and Th. 6, p. 19—21).

Hence, we have

$$(2.2)\qquad \frac{T_n^*(x)}{\mu_n} = \bar{T}_n^*(x) - \frac{1}{P_n}\sum_{k=0}^{n-1} P_k T_k^*(x)\,\Delta\frac{1}{\mu_k} - \frac{1}{P_n}\sum_{k=0}^{n} p_k \sum_{\nu=0}^{n-1} s_\nu^*(x)\,\Delta\frac{1}{\mu_\nu}.$$

It is easy to verify that in virtue of (1.3), both series

$$(2.3)\qquad \sum_{\nu=0}^{\infty} |s_\nu^*(x)|\,\Delta\frac{1}{\mu_\nu}\,;\ \sum_{\nu=0}^{\infty} |T_\nu^*(x)|\,\Delta\frac{1}{\mu_\nu}$$

are convergent almost everywhere.

Applying to series (2.3) the well-known Kronecker theorem, we obtain the following relations

$$(2.4)\qquad \frac{1}{P_n}\sum_{k=0}^{n-1} P_k T_k^*(x)\,\Delta\frac{1}{\mu_k} \doteq o(1), \text{ when } n \to \infty,$$

$$(2.5)\qquad \frac{1}{P_n}\sum_{k=0}^{n} p_k \sum_{\nu=0}^{k-1} s_\nu^*(x)\,\Delta\frac{1}{\mu_\nu} \doteq O(1), \text{ when } n \to \infty.$$

Assuming $\mu_n = \log\log P_n$ and applying the Zygmund theorem ([5], Th. I, p. 296) we conclude that

$$\bar{T}_n^*(x) \doteq O(1), \text{ when } n \to \infty.$$

This fact and (2.2), (2.3), (2.4), (2.5) imply

$$T_n^*(x) \doteq O(\log\log P_n), \text{ when } n \to \infty.$$

Thus, from Lemma 1 it follows that

$$T_n(x) \doteq o(\log\log P_n), \text{ when } n \to \infty,$$

which completes the proof of Theorem 1.

THEOREM 2. *Let us assume $\{\nu(n)\}$ to be an increasing sequence of positive numbers tending to infinity and satisfying the condition*

$$(2.6)\qquad \nu(n) = o(\log\log P_n), \text{ when } n \to \infty.$$

Then there exists a numerical sequence $\{c_n\}$ and such an ON-system $\{\varphi_n(x)\}$ that $\{c_n\} \in l^2$ and

$$\overline{\lim_{n\to\infty}}\frac{1}{\nu(n)}\left|\frac{1}{P_n}\sum_{k=0}^{n} p_k \sum_{\nu=0}^{k} c_\nu \Phi_\nu(x)\right| = \infty,$$

everywhere in $\langle 0,1 \rangle$.

Proof. Since $\nu(n) = o(\log\log P_n)$ and $\nu(n) \uparrow \infty$, the above mentioned theorem of Zygmund implies the existence of a sequence $\{b_n\}$ and such an ON-system

$\{\Phi_n(x)\}$ that $\sum_{n=0}^{\infty} b_n^2 v^2(n) < \infty$ and the series $\sum_{n=0}^{\infty} b_n \Phi_n(x)$ is non-summable by the method $(\bar{N}, p_n)$ at any point of the interval $\langle 0,1 \rangle$.

Assuming in the formulas (1.4) and (2.2)

$$a_n = b_n v(n),\ \mu_n = \lambda_n v(n),\ \varphi_n(x) = \Phi_n(x),$$

we find that

$$(2.7)\quad \frac{T_n^*(x)}{v(n)} = \lambda_n \left[\bar{T}_n^x(x) - \frac{1}{P_n} \sum_{k=0}^{n-1} P_k T_k^*(x) \Delta \frac{1}{\mu_k} - \frac{1}{P_n} \sum_{k=0}^{n} p_k \sum_{v=0}^{k-1} s_v^*(x) \Delta \frac{1}{\mu_v} \right].$$

Let $\{\lambda_n\}$ denote a sequence of positive numbers increasing to infinity and such that $\sum_{n=0}^{\infty} b_n^2 \lambda_n^2 v^2(n) < \infty$.

In virtue of (2.4), the last but one expression in square brackets tends to zero almost everywhere, whereas the last one is bounded almost everywhere by (2.5). From the definition of the sequence $\{b_n\}$ and of the system $\{\Phi_n(x)\}$, it follows now, that the sequence $\{\bar{T}_n^*(x)\}$ is divergent almost everywhere.

Since, $\{\lambda_n\}$ tends to $+\infty$, we find that

$$(2.8)\quad \overline{\lim_{n \to \infty}} \frac{|T_n^*(x)|}{v(n)} = +\infty$$

almost everywhere in $\langle 0,1 \rangle$.

Now, in order to prove that the last formula holds at any point of the interval $\langle 0,1 \rangle$, we extend the sequence $\{P_n\}$ by a linear interpolation to the strictly increasing function $P(x)$, which for positive integers n takes the values $P(n) = P_n$. We denote the function inverse to function $P(x)$ by $R(x)$.

Further, we denote by E the set of all $x \in \langle 0,1 \rangle$, such that (2.8) does not hold at the point x. The functions of the system ON $\{\Phi_n(x)\}$ will be modified as follows: if $x \in E$, then we put

$$\Phi_{[R(2^n)]}(x) = \frac{1}{b_{[R(2^n)]}} \text{ for } N_{k-1} < n < N_k\ (k = 1, 2, \ldots)^*)$$

and

$$b_n = 0,\ \Phi_n(x) = \varphi_{N_k}(x)$$

for the remaining indices n, successively, wherein $\{N_k\}$, with $N_0 = 0$, denotes the sequence and $\{\varphi_n(x)\}$ the ON-system, defined in the proof of the Menchoff theorem (see e.g. [2], p. 195, sec. 5.3.7).

Now, let us suppose that $\lambda_n \geqslant 2$, $v(n) \geqslant 1$ $(n = 1, 2, \ldots)$, which does not restrict the generality of our proof.

Evidently, we have

$$\frac{T_n^*(x)}{v(n)} > \frac{1}{v(n)} \cdot \frac{1}{P_n} \sum_{k=n_0}^{n} p_k \sum_{\mu=1}^{s} \sum_{\substack{N_{\mu-1} < m < N_\mu \\ 0 < m \leqslant m_0}} \lambda_{[R(2^m)]}\, v_{[R(2^m)]},$$

*) $[x]$ denotes the integral part of x.

where n_0 denotes the least positive integer such that

$$\log\log P_{n_0} \geqslant 1 \text{ and } n_0 \geqslant [R(2^{2^{N_3}})] + 1,$$

m_0 — the greatest positive integer such that $[R(2^{2^{m_0}})] \leqslant k$, and s is defined by the inequalities

$$[R(2^{2^{N_s - 1}})] + 1 \leqslant k < [R(2^{2^{N_s}})].$$

Denoting by p the number of terms in the expression

$$\sum_{\mu=1}^{s} \sum_{\substack{N_\mu - 1 < m < N_\mu \\ 0 < m \leqslant m_0}} \lambda_{[R(2^{2^m})]}\, v_{[R(2^{2^m})]}$$

and taking into account that

$$1 + m_0 > \log_2\log_2 P_k > N_{s-1} > 2s\,;\; p = m_0 - (s-1) > \frac{1}{2}\log_2\log_2 P_k > \frac{1}{2}\log\log P_k,$$

we can write

$$\frac{T_n^*(x)}{v(n)} > \frac{1}{v(n)}\frac{1}{P_n}\sum_{k=n_0}^{n} p_k \log\log P_k = o(1) + \frac{\log\log P_k}{v(n)} +$$

$$+ \frac{1}{v(n)\,P_n}\sum_{n-n_0}^{n-1} P_k\, \Delta(\log\log P_k).$$

The last but one expression tends to infinity by (2.6) and the last tends to zero, for

$$\frac{1}{P_n}\sum_{k=n_0}^{n-1} P_k(\log\log P_{k+1} - \log\log P_k) =$$

$$= \frac{1}{P_n}\sum_{k=n_0}^{n-1} e^{\log P_k} \int_{\log P_k}^{\log P_{k+1}} \frac{du}{u} < \frac{1}{P_n}\int_{\log P_{n_0}}^{\log P_n} e^u\, du = O(1).$$

Finally, we have

$$\frac{|T_n^*(x)|}{v(n)} \to \infty, \text{ for } x \in E \text{ and } n \to \infty,$$

then Theorem 2 results, putting $c_n = b_n\, \lambda_n\, v(n)$.

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REFERENCES

[1] G. H. Hardy, *Divergent Series*, Oxford, 1949.

[2] S. Kaczmarz and H. Steinhaus, *Theory of orthogonal series* [in Russian], Moscow, 1958.

[3] J. Meder, *On the summability almost everywhere of orthonormal series by the method of first logarithmic means*, Rozpr. Matem., **17** (1959), 1—34.

[4] K. Tandori, *Über die orthogonalen Funktionen I*, Acta Sci. Math., **18** (1957), 57—130.

[5] A. Zygmund, *Sur la sommation des séries de fonctions orthogonales*, Bull. Intern. de l'Acad. Sci. et Lettr., Classe A, (1927), 295—308.

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MATHEMATICS

On Two-to-One Continuous Functions

by

J. MIODUSZEWSKI

Presented by K. BORSUK on December 20, 1960

Introduction

In this note a summary is given of the general investigation of the so-called *k-to-one functions*, i.e. continuous functions which assume every value in exactly k points. The general assumption concerning the space of arguments X of such a function f is that of local compactness and separability. In order to exclude the triviality, we assume that $f(X)$ is a subspace of an T_2-space (or Hausdorff space) Y. The most interesting case is that of $k = 2$, i.e. of the *two-to-one functions*. In this case there is a very interesting problem of the existence of such functions (see [1]—[6]). The most important result due to Civin [3] asserts that there do not exist two-to-one functions on n-cells for $n \leqslant 3$. The problem which remains open is that of the existence of the two-to-one functions on n-cells for $n > 3$.

The investigation of the two-to-one functions is, in a natural manner, equivalent to the investigation of some involution φ, where $\varphi(x)$ is the element of $f^{-1}f(x)$, which is different from x. Involution φ is, in general, discontinuous, but it is, in a sense, *semicontinuous* namely

$$\operatorname{Ls}_{\xi \to x} \varphi(\xi) \subset \varphi(x) \cup x \cup p,$$

where p is the point adjoined to X by one-point compactification of X. In [3] Civin has showed that the investigation of φ on compact manifolds or if f is closed on locally compact manifolds, is equivalent to the investigation of a continuous involution. In this note we do not assume that X is a manifold and, if X is non compact, that f is closed. The most general problem, which is considered here, is that of the behaviour of φ on the closures of the so-called pseudo-neighbourhoods of euclidean points or points which in a sense are pseudo-euclidean. The main theorem asserts that the investigation of φ on such subsets of X is equivalent to the investigation of a continuous involution. Using this theorem it is possible to investigate two-to-one continuous functions on some spaces which are not-manifold.

The method used in proofs is that of multi-valued upper semicontinuous functions. It is known that such functions are of the 1-st Baire class with respect to the topology induced by Hausdorff metric in the space of compact subsets of topological space in question (see [7], II).

The general properties of involution φ

Denote by X_φ the set of continuity points of φ. Then (see also Roberts [2]).

(1) *X_φ is a dense and open subset of X.*

Consider the set $\psi^*(x) = \operatorname{Ls}_{\substack{\xi \to x \\ \xi \in X_\varphi}} \varphi(\xi)$. The point x of discontinuity of φ is said to be *weakly essential* if $\psi^*(x) = x \cup \varphi(x)$. It is said to be *strongly essential* if $\psi^*(x)$ contains p. We shall write in the first case that *x is a* WED-*point of φ* or $x \in \mathrm{WED}(\varphi)$, and in the second case that *x is a* SED-*point of φ* or $x \in \mathrm{SED}(\varphi)$.

A point $x \in X$ is said to be *pseudo-euclidean* if there exists a neighbourhood H of x in X such that the closure of the component of x in H is a euclidean solid sphere. We shall call such components H the *euclidean pseudo-neighbourhoods*.

The main theorem asserts

THEOREM 1. *A pseudo-euclidean point $x \in X$ cannot be a weakly essential point of $\varphi | X$, where X is the closure of an euclidean pseudo-neighbourhood of x in X.*

In the proof we use a theorem of Newman [8] concerning continuous involutions on closures of subdomains of compact manifolds.

As a corollary we obtain

THEOREM 2. *Let $R \subset X$ be a manifold having the following property*:

(i) *if $x \in R$ then there exists a pseudo-neighbourhood of x in X which is a neighbourhood of x in R simultanously.*

If $\varphi | R$ is without SED-*points then the function $\widetilde{\varphi}: R \to X$ given by*

(ii) $$\widetilde{\varphi}(\xi) = \begin{cases} \varphi(\xi) & \text{if } \xi \in R_\varphi \\ \xi & \text{if } \xi \in R - R_\varphi . \end{cases}$$

is continuous and one-to-one.

Furthermore, if $\widetilde{\varphi}(R) \subset R$, then $\widetilde{\varphi}(R) = R$, $\widetilde{\varphi}(R_\varphi) = \varphi(R_\varphi) = R_\varphi$, $\widetilde{\varphi}(R - R_\varphi) = R - R_\varphi$, and $\widetilde{\varphi}$ is an involution on R and it cannot be the identity on open subsets of R.

Corollaries for some compact space

The theorem of Civin concerning the behaviour of φ on manifolds is a special case of Theorem 2. As a corollary (which is not possible to obtain from Civin's theorem) we obtain from Theorem 2 a simple proof of the following theorem of Harrold [4] and Gilbert [5]:

(2) *There do not exist two-to-one functions on graphs of which the Euler-Poincaré characteristic is odd.*

As an example of application of Theorem 2 to non locally connected continua we prove that

(3) *There do not exist two-to-one functions on the irreductible continuum $\mathfrak{N}_V$* (see for description [9], p. 570) *and on the indecomposable continuum $\mathfrak{I}_0$* (see for description [7], II, p. 143).

All points of both continua are pseudo-euclidean. There is no method for the investigation of the two-to-one functions on spaces whose local structure is "worse" than that of $\mathfrak{N}_V$ and $\mathfrak{B}_0$.

We give an example of a plane simply connected domain G whose boundary is an irreductible cut of the plane into two domains G and $E^2 - \bar{G}$, and such that

(4) *There exist two-to-one function on* $\bar{G}$.

This is in contrast to results of Roberts [2] concerning the non-existence of two-to-one functions on 2-cells.

A general result concerning simply connected plane domains G asserts that

(5) *The involution* φ *on* $\dot{G}$ *induces a continuous involution on the space consisting of boundary elements and points of* $\bar{G}$.

The case of locally compact manifolds

According to Theorem 1 the involution φ has no WED-point if $\mathcal{X}$ is a manifold. However, if $\mathcal{X}$ is only a locally compact manifold, then there can exist involutions φ (induced by two-to-one functions with SED-points. Consider the function $\tilde{\varphi}: \mathcal{X} - - \mathrm{SED}(\varphi) \to \mathcal{X}$ defined by

$$\tilde{\varphi}(\xi) = \begin{cases} \varphi(\xi) & \text{if} \quad \xi \in \mathcal{X}_\varphi \\ \xi & \text{if} \quad \xi \in \mathcal{X} - \mathrm{SED}(\varphi) - \mathcal{X}_\varphi \end{cases}. \tag{6}$$

The following property of $\tilde{\varphi}$ is an obvious consequence of Theorem 2.

(7) *If* $\mathcal{X}$ *is a (locally compact) manifold without boundary then* $\tilde{\varphi}$ *is a continuous involution on* $\mathcal{X} - \mathrm{SED}(\varphi)$, *and it cannot be the identity on open subsets of* $\mathcal{X} - \mathrm{SED}(\varphi)$; *moreover* $\tilde{\varphi}(\mathcal{X}_\varphi) = \varphi(\mathcal{X}_\varphi) = \mathcal{X}_\varphi$ *and* $\tilde{\varphi}(\mathcal{X} - \mathcal{X}_\varphi) = \mathcal{X} - \mathcal{X}_\varphi$.

The theorem which follows is the most important one in the investigation of the structure of SED (φ). Every homeomorphic image of the closed interval $0 \leqslant t \leqslant 1$, given by a homeomorphism h such that $h(0) = x$ and $h(t) \in \mathcal{X} - \mathrm{SED}(\varphi)$ for $t \neq 0$, is called a *path to the point* x. A point $x \in \mathrm{SED}(\varphi)$ is said to be *strongly accessible* from $\mathcal{X} - \mathrm{SED}(\varphi)$ if there exists a path to x such that

$$\lim_{n \to 0} \tilde{\varphi}\, h(t) = p.$$

THEOREM 3. *If* $\mathcal{X}$ *is a manifold,* x *is a* SED*-point of* φ, *and* U *is an open neighbourhood of* x *in* $\mathcal{X}$, *then there exists a* SED*-point of* φ *in* U, *which is strongly accessible from* $\mathcal{X} - \mathrm{SED}(\varphi)$.

If $\mathcal{X}$ is the straight line then from theorem 3 it follows that φ has at most two SED-points. From this corollary we obtain in an elementary way, that

(8) *There do not exist two-to-one fonctions on the straight line.*

From (8) it follows that

(9) *If* f *is a two-to-one function on the plane, then the associated semicontinuous involution* φ *has* SED*-points.*

From Theorem 3 it follows that

(10) *If* $\mathcal{X}$ *is the half-straight line, then involution* φ *has exactly one* SED*-point.*

Denote this point by x_0. Let the half-straight line be represented by the interval $q \leqslant x < p$. Let $X_1 = (q\, x_0)$ and $X_2 = (x_0\, p)$ be intervals without ends. Then

(11) $\widetilde{q}(X_i) = X_{j(i)}$, *where* $i, j(i) = 1$ *or* 2; *furthermore if* $i_1 \neq i_2$, *then* $j(i_1) \neq j(i_2)$.

From (11) it follows that

(12) *There exist exactly three kinds of two-to-one functions on the half-straight line described by the following relations*:

(i) $x_0 \neq q$ *and* $\widetilde{\varphi}(X_1) = X_1$,

(ii) $x_0 \neq q$ *and* $\widetilde{\varphi}(X_1) = X_2$,

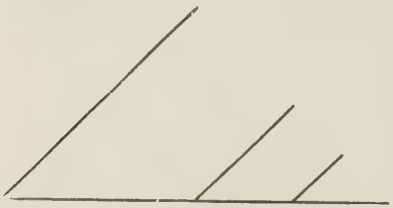
(iii) $x_0 = q$.

The existence of two-to-one functions on euclidean spaces E^n for $n \geqslant 2$

This problem was raised by Civin [3]. In contrast to n-cells for $n \leqslant 3$, we have

THEOREM 4. *There exist two-to-one functions on euclidean spaces* E^n *for* $n \geqslant 2$.

The construction of such functions is as follows. Consider E^n as $S^n - C$, where C is a continuum such that there exist two-to-one function on it. There exist continua C having the above property, for example the annexed dendrite and the closure of the domain G of the assertion (4). Denote by C^- the image of C by the antipodism and assume that $C \cap C^- = 0$. In order to define two-to-one function on $S^n - C$, we define the following involution λ on $S^n - C$. We set $\lambda(x) = x^-$ for $x \in S^n - C - C^-$ and $\lambda(x) = (\varphi(x))^-$ for $x \in C^-$, where φ is the semicontinuous involution on C induced by two-to-one function on C. It is easy to verify that involution λ induces a function on $S^n - C$, which is two-to-one and continuous.



The details of constructions and proofs of theorems will be given in Rozprawy Matematyczne.

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REFERENCES

[1] O. G. Harrold, *The non-existence of a certain type of continuous transformations*, Duke Math. Journal **5** (1939), 789—793.

[2] J. H. Roberts, *Two-to-one transformations*, Duke Math. Journal **6** (1940), 256—262.

[3] P. Civin, *Two-to-one mappings of manifolds*, Duke Math. Journal **10** (1943), 49—57.

[4] O. G. Harrold, *Exactly* (k, l) *transformations on connected linear graphs*, American Journal of Math. **62** (1940), 823—834.

[5] Venable Martin and J. H. Roberts, *Two-to-one transformations on 2-manifolds*, Trans. Amer. Math. Soc. **49** (1941), 1—17.

[6] P. W. Gilbert, *n-to-one mappings on linear graphs*, Duke Math. Journal **9** (1942), 475—486.

[7] C. Kuratowski, *Topologie*, I (1948) and II (1950), Warszawa—Wrocław.

[8] H. M. A. Newman, *A theorem on periodic transformations of spaces*, The Quarterly Journal of Math., Oxford Series, **2** (1931), 1—8.

[9] B. Knaster, *Un continu irréductible à décomposition continue en tranches*, Fund. Math, **25** (1935), 568—577.

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MATHEMATICS

Algebraic Properties of Linear Derivative Equations in Linear Spaces

by

R. BITTNER

Presented by S. MAZUR on December 23, 1960

1. Introduction

The well-known theorem about fundamental solutions of linear homogeneous differential equations

$$a_n(t)\,x^{(n)}(t)+a_{n-1}(t)\,x^{(n-1)}(t)+\ldots+a_0(t)\,x(t)=0,$$

where functions $a_n(t), \ldots, a_0(t)$ are continuous, states that there exist n linear independent solutions $x_1(t), \ldots, x_n(t)$ such that all solutions are of the form

$$x(t)=C_1\,x_1(t)+\ldots+C_n\,x_n(t),$$

where $C_1, \ldots, C_n$ are constant numbers. Similar theorem is true for a system of linear differential equations.

Let us consider, as in [2] and [3], a derivative S that is a linear operation S from C^1 onto C^0 ($C^1 \subset C^0$), where C^1 and C^0 are linear spaces over the same field I. The elements $c \in K(S)$ such that $Sc=0$ are called constants. An operation T such that $STx=x$ for $x \in C^0$ is called integral. The operation $sx \quad x-TSx$ is called limit condition.

The operations $X_1, \ldots, X_n$ defined on the same linear set K of constants will be called operationally independent if and only if

$$X_1\,c_1+\ldots+X_n\,c_n=0 \text{ implies } c_1=\ldots=c_n=0.$$

This paper treats the problem of operational independence of solving operations of an abstract linear homogeneous derivative equation

$$A_n\,S^n\,x+A_{n-1}\,S^{n-1}\,x+\ldots+A_0\,x=0, \tag{1}$$

where $A_n, A_{n-1}, \ldots, A_0$ are endomorphisms of C non-commutative in general with S. It can be proved that if Eq. (1) with limit conditions

$$s\,S^i\,x=0, \quad i=0,1,\ldots,n-1$$

has only trivial solution $x=0$, then all solutions x of Eq. (1) are of the form

$$x=X_0\,c_0+\ldots+X_{n-1}\,c_{n-1}, \tag{2}$$

where $X_0 c_0, \ldots, X_{n-1} c_{n-1}$ are n known operationally independent solving operations, i.e. having the values in the space of solutions, and $c_0, \ldots, c_{n-1}$ are constants. These solutions are in general in a space $C_h^0(\Pi)$ of hyper-results (see par. **2**). The number of classic operationally independent solving operations is $k \leqslant n$. The sufficient and necessary conditions for operational independence are given.

2. Results and hyper-results

As in [2], a result $\frac{f}{W(T)} \in C^0(\Pi)$ will be a formal fraction of an element $f \in C^0$ and a polynomial $W(T) \in \Pi$, where Π is a commutative group of polynomials $W(T)$ such that $W(T)x = 0$ implies $x = 0$. The equality of these fractions, addition and multiplication by a number are the same as the operations on rational numbers.

We shall also consider hyper-results $x_0 + T\frac{f}{W(T)} \in C_h^0(\Pi)$ that is pairs $\left[x_0, \frac{f}{W(T)}\right]$, where $x_0 \in K(S)$, $\frac{f}{W(T)} \in C^0(\Pi)$, with usual equality, addition and multiplication of pairs by numbers. In this way we obtain a linear space. Pairs $[x_0, 0]$ can be identified with x_0. Instead of $\left[x_0, \frac{f}{W(T)}\right]$ we shall write $x_0 + T\frac{f}{W(T)}$.

The hyper-result $x_0 + T\frac{f}{W(T)}$ is the result, if $\frac{f}{W(T)} \in C^0$ or if $W(T)$ commute with T.

We can extend the derivative S and the limit condition s on the hyper-results and the integral T on the results by formulas

$$S[x_0, \xi] = \xi, \quad T\xi = [0, \xi], \quad s[x_0, \xi] = [x_0, 0].$$

We have then $ST\xi = \xi$ for $\xi \in C^0(\Pi)$, $sx = x - TSx$ for $x \in C_h^0(\Pi)$.

The equation

$$A_1 Sx + A_0 x = f, \tag{3}$$

where $f \in C^0$ and A_1, A_0 are endomorphisms of C^0 with limit condition

$$sx = x_0 \tag{4}$$

is equivalent to the equation

$$(A_1 + A_0 T) Sx = f - A_0 x_0$$

If the homogeneous Eq. (3) with homogeneous limit conditions (4) has only trivial solution we can take as a semigroup Π the semigroup of polynomials

$$W(T) = \alpha (A_1 + A_0 T)^k, \qquad \alpha \in \Gamma, \ k = 0, 1, 2, \ldots$$

and we obtain x as a hyper-result

$$x = x_0 + T\frac{f - A_0 x_0}{A_1 + A_0 T} = X_0 x_0 + Xf. \tag{5}$$

If $\frac{f - A_0 x_0}{A_1 + A_0 T} \in C^0$ then $x \in C^1$ and x is the classic solution of Eq. (3) with limit condition (4). In general, this condition is not satisfied and we shall say that in this case x is the generalised solution of Eq. (3) with limit condition (4).

3. A system of linear derivative equations of first order

Let us consider a system of n linear derivative equations of first order with n unknowns $x_1, \ldots, x_n$:

$$A_i^1 S x_i + \sum_{j=1}^{n} A_{ij}^0 x_j = f_i, \qquad i = 1, \ldots, n \tag{6}$$

with limit conditions

$$s x_i = x_{i0}, \qquad i = 1, \ldots, n. \tag{7}$$

We can consider this system as one linear derivative equation of first order

$$\mathfrak{A}_1 \mathfrak{S} \mathfrak{x} + \mathfrak{A}_0 \mathfrak{x} = \mathfrak{f}$$

$$\mathfrak{s} \mathfrak{x} = \mathfrak{x}_0,$$

where the derivative $\mathfrak{S} = [\delta_{ij} S]$, the limit condition $\mathfrak{s} = [\delta_{ij} s]$ and the endomorphisms are $\mathfrak{A}_1 = [\delta_{ij} A_i^1]$, $\mathfrak{A}_0 = [A_{ij}^0]$. As elements $\mathfrak{x}$ and $\mathfrak{f}$ we take $\mathfrak{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathfrak{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$. The compositions of linear operations $\mathfrak{A}_1, \mathfrak{S}, \mathfrak{A}_0$, and of element $\mathfrak{x}$ are the usual multiplications of matrices. $\mathfrak{K} = K(\mathfrak{S})$ is the set of constants $\mathfrak{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, where $c_i \in K(S)$.

In this way, assuming that the homogeneous equation (6) with homogeneous limit conditions (7) has only trivial solution $x_1 = \ldots = x_n = 0$ and calling n operationally independent solving operations $X_1 x_{10}, \ldots, X_n x_{n0}$ of homogeneous Eq. (6) fundamental system we obtain the following theorems:

THEOREM 1. *There exist n solving operations* $\mathfrak{X}_1 x_{10}, \ldots, \mathfrak{X}_n x_{n0}$ *of homogeneous* Eq. (6) *such that they form a fundamental system. Each solution* x *of homogeneous* Eq. (6) *has the form*

$$\mathfrak{x} = \mathfrak{X}_1 x_{10} + \ldots + \mathfrak{X}_n x_{n0}$$

and

$$s x_i = x_{i0}.$$

These solving operations are called normal fundamental solving operations.

THEOREM 2. *If* $\bar{\mathfrak{X}}_1 \bar{\mathfrak{x}}_1, \ldots, \bar{\mathfrak{X}}_n \bar{\mathfrak{x}}_n$ *form the fundamental system of solving operations then*

$$\bar{\mathfrak{X}}_1 \bar{\mathfrak{x}}_1 + \ldots + \bar{\mathfrak{X}}_n \bar{\mathfrak{x}}_n = \mathfrak{X}_1 \mathfrak{P}_1 \bar{\mathfrak{x}}_1 + \ldots + \mathfrak{X}_n \mathfrak{P}_n \bar{\mathfrak{x}}_n,$$

where $\mathfrak{P}_i$ *are invertible linear operations on the set* $\mathfrak{K}$ *of constants.*

THEOREM 3. *The n solving operations* $\mathfrak{X}_1 \mathfrak{x}_1, \ldots, \mathfrak{X}_n \mathfrak{x}_n$ *form the fundamental system if and only if the equation*

$$s\bar{\mathfrak{X}}_1 \bar{\mathfrak{x}}_1 + \ldots + s\bar{\mathfrak{X}}_n \bar{\mathfrak{x}}_n = 0$$

has only trivial solution $\bar{\mathfrak{x}}_1 = \ldots = \bar{\mathfrak{x}}_n = 0$.

THEOREM 4. *If any fundamental system of solving operations* $\bar{\mathfrak{X}}_1 \mathfrak{x}_1, \ldots, \mathfrak{X}_n \mathfrak{x}_n$ *and any solution* $\mathfrak{x}$ *of the* Eq. (6) *are known, then the solution* $\mathfrak{x}$ of Eq. (6) *with limit conditions* (7) *has the form*

$$\mathfrak{x} = \mathfrak{X}_1 \mathfrak{P}_1^{-1} (x_{10} - s x_1) + \ldots + \bar{\mathfrak{X}}_n \mathfrak{P}_n^{-1} (x_{n0} - s x_n) + \bar{\mathfrak{x}}.$$

4. The linear derivative equation of order n

The linear derivative equation of order n

$$A_n S^n x + A_{n-1} S^{n-1} x + \ldots + A_0 x = f \tag{8}$$

with limit conditions

$$sS^i x = x_{i0}, \quad i = 0, 1, \ldots, n-1 \tag{9}$$

can be treated as a system of linear derivative equations of first order. In this way assuming that the homogeneous Eq. (8) with homogeneous limit conditions (9) has only trivial solution $x = 0$ and calling n operational independent solving operations $X_0 x_{00}, \ldots, X_{n-1,0}$ of homogeneous Eq. (8) the fundamental system we obtain the following theorem:

THEOREM 5. *There exist n solving operations* $X_0 x_{00}, \ldots, X_{n-1} x_{n-1,0}$ *of homogeneous* Eq. (8) *such that they form a fundamental system, each solution x of homogeneous* Eq. (8) *has the form*

$$x = X_0 x_{00} + \ldots + X_{n-1} x_{n-1,0}$$

and

$$sS^i X_j x_{j0} = \delta_j^i x_{j0}, \quad i, j = 0, 1, \ldots, n-1.$$

These solving operations are called normal fundamental solving operations.

THEOREM 6. *If* $\bar{X}_0 x_{00}, \ldots, X_{n-1} x_{n-1,0}$ *form the fundamental system of solving operations then*

$$\bar{X}_0 \bar{x}_{00} + \ldots + \bar{X}_{n-1} \bar{x}_{n-10} = X_0 R_0 \bar{x}_{00} + \ldots + X_{n-1} R_{n-1} \bar{x}_{n-1,0},$$

where R_i *are invertible linear operations on the set of elements*

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}, \quad c_i \in K(S^{i+1}).$$

THEOREM 7. *The n solving operations $X_0\, x_{00}, \ldots, \overline{X}_{n-1}\, x_{n-10}$ form the fundamental system if and only if the system of equations*

$$\sum_{k=0}^{n-1} sS^i\, X_k\, x_{k0} = 0$$

has only trivial solution $\bar{x}_{00} = \ldots = \bar{x}_{n-1,\,0} = 0$.

THEOREM 8. *If any fundamental system of solving operations $\overline{X}_0\, x_{00}, \ldots, X_{n-1}\, x_{n-1,\,0}$ and any solution $\bar{x}$ of the* Eq. (8) *are known then, the solution x of* Eq. (8) *with limit conditions* (9) *has the form*

$$x = \overline{X}_0\, R_0^{-1}\,(x_{00} - s\bar{x}) + \ldots + \overline{X}_{n-1}\, R_{n-1}^{-1}\,(x_{n-1,\,0} - sS^{n-1}\,\bar{x})\,.$$

5. Partial derivative equations of order n

We can consider in general derivatives $S_1, \ldots, S_m$ and m integrals $T_1, \ldots, T_m$ such that

$$S_i\, T_j = I,\ S_i\, S_j = S_j\, S_i,\ T_i\, T_j = T_j\, T_i \quad \text{for} \quad i, j = 1, \ldots, m.$$

We have then m limit conditions $s_1, \ldots, s_m$, where

$$s_i = I - T_i\, S_i \quad \text{for} \quad i = 1, \ldots, m\,.$$

The partial derivative equation has the form

$$\sum_{(i_1, \ldots, i_m)} A_{i_1 \ldots i_m}\, S_1^{i_1} \ldots S_m^{i_m}\, x = f. \tag{10}$$

We shall say that Eq. (10) has order $N = \max(i_1, \ldots, i_m)$. Let be $u_k = \max i_k$ $N_1 = u_1 + \ldots + u_m$. For the equation of order N we shall give the limit conditions

$$s_i\, S_j^k\, x = x_{ijk}\,, \quad i, j = 1, \ldots, m, \quad k = 0, 1, \ldots, N-1. \tag{11}$$

We take a new derivative and a new limit condition

$$S = S_1 \ldots S_m, \quad s = I - T_1 \ldots T_m\, S_1 \ldots S_m$$

so that Eq. (10) is by the derivative S an ordinary derivative equation of order N and limit conditions (11) give the limit condition for the new equation.

In the following we shall assume that the homogeneous Eq. (10) with homogeneous limit conditions (11) has only trivial solution $x = 0$. Calling operational independent solving operations $X_{ijk}\, x_{ijk}$ of homogeneous Eq. (10) the fundamental system, we obtain following theorems:

THEOREM 9. *There exist N_1 solving operations $X_{ijk}\, x_{ijk}$ of homogeneous* Eq. (10) *such that they form a fundamental system, each solution x of homogeneous* Eq. (10) *has the form*

$$x = \sum_{\substack{i=j=l=1,2,\ldots,m \\ k=i_l=0,1,\ldots,u_l-1}} X_{ijk}\, x_{ijk}$$

dan

$$s_{i_1}\, S_{j_1}^{k_1}\, X_{ijk}\, x_{ijk} = \delta_{i_1}^{i}\, \delta_{k_1}^{k}\, \delta_{j_1}^{j}\, x_{ijk}\,.$$

These solving operations are called normal fundamental solving operations.

THEOREM 10. *If $\bar{X}_{ijk}\,\bar{x}_{ijk}$ form the fundamental system of N_1 solving operations, then*

$$\sum_{i,j,k} \bar{X}_{ijk}\,\bar{x}_{ijk} = \sum_{i,j,k} X_{ijk}\,R_{ijk}\,\bar{x}_{ijk},$$

where R_{ijk} are invertible linear operations on the set of elements

$$x_{ijk} = \begin{vmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{vmatrix}, \quad c_i \in K(S_1^{i+1} \dots S_n^{i+1}).$$

THEOREM 11. *The N_1 solving operations $X_{ijk}\,x_{ijk}$ form the fundamental system if and only if the system of equations*

$$\sum_{i,j,k} sS^l\,\bar{X}_{ijk}\,\bar{x}_{ijk} = 0, \qquad l = 0, 1, \dots, N-1$$

has only trivial solution.

THEOREM 12. *If any fundamental system of N_1 solving operations $X_{ijk}\,x_{ijk}$ N_1 and any solution x of the* Eq. (12) *are known, then the solution x* Eq. (12) *with limit conditions* (13) *has the form*

$$x = \sum_{i,j,k} \bar{X}_{ijk}\,R_{ijk}^{-1}\,(x_{ijk} - s_i\,S_j^k\,\bar{x}).$$

A. In Mikusiński's operational calculus the equation $x''(\lambda) + s^2\,x(\lambda) = 0$ has only trivial solution (see paper [6]). Similarly the system of equations $x'(\lambda) = y(\lambda)$, $y'(\lambda) = -s^2\,x(\lambda)$ has only trivial solution so that the number of classic solutions can be less than the order of equation. Our theory gives the result of Mikusiński [6] about extensions of solutions as a special case.

B. The normal fundamental solving operations of the equation of string

$$\frac{\partial^2 u}{\partial t^2} = a^2\,\frac{\partial^2 u}{\partial x^2} \tag{12}$$

(where $u(x, 0)$, $u_t(x, 0)$ are given) are operations

$$\frac{\varphi(x-at) + \varphi(x+at)}{2}, \quad \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\,d\xi$$

on the constants $\{\varphi(x)\}$ and $\{\psi(x)\}$ for the derivative $S = \frac{\partial}{\partial t}$. The limit condition is $s\{u(x,t)\} = \{u(x,0)\}$, and $\frac{\partial^2}{\partial x^2} = A$ is an endomorphism of space C^0 of smooth functions or of distributions.

C. For a partial differential equation of the second degree and hyperbolic type which can be treated in canonical form as a partial derivative equation of first order $\left(S_1 = \frac{\partial}{\partial x}, S_2 = \frac{\partial}{\partial y}\right)$ different limit conditions such as Cauchy's, Darboux's and Picard's can be given (see example *C* in paper [3] about generalization of Szmydt condition). If *s* is the given limit condition, Cauchy's for example, then the system of solving operations satisfying Cauchy's condition is the normal fundamental system, and the system satisfying Darboux conditions is the fundamental one and vice versa.

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REFERENCES

[1] S. Bellert, *On foundations of operational-calculus*, Bull. Acad. Polon. Sci., Classe III, **5** (1957), 855—858.

[2] R. Bittner, *Operational calculus in linear spaces*, Studia Math., **20** (1961), 1—18.

[3] — , *About a new definition of polynomials*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **9** (1961),

[4] J. Mikusiński, *Sur les théoremes d'unicité et le nombre de solutions linéairement indépendantes*, Studia Mathematica XVI, 2 (1957), 95—98.

[5] — , *Sur l'espace linéaire avec dérivation*, Studia Mathem., **16** (1957), 113—123.

[6] — , *Extensions de l'espace linéaire avec dérivation*, Studia Mathem., **16** (1957), 156—172.

[7] W. Słowikowski, *A generalisation of the theory of distributions*, Bull. Acad. Polon. Sci. Classe III, **3** (1955), 3—6.

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MATHEMATICS

Algebraic Properties of Linear Derivative Equations in Algebras

by

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Presented by S. MAZUR on December 23, 1960

1. Introduction

Let us consider, as in paper [4], a derivative S, that is a linear operation S from C^1 onto C^0 ($C^1 \subset C^0$), and let us suppose that C^1 and C^0 are commutative algebras over the same field Γ. Multiplication $x \cdot y$ satisfies conditions

$$S(x \cdot y) = Sx \cdot y + x \cdot Sy, \quad s(x \cdot y) = sx \cdot sy \quad \text{for} \quad x, y \in C^1, \tag{1}$$

where $sx = x - TSx$ is the limit condition and T satisfies the condition $STx = x$. Of course, all theorems of paper [4] are valid. If solving operations defined in [4] are multiplications by known elements, then the sufficient and necessary conditions for operational independence result in Wroński's criteria. For equations with number coefficients the method of variation of constants can be applied.

2

If C^0 is a commutative algebra with multiplication $x \cdot y$ satisfying (1), then the space $\mathfrak{C}^0$ of elements $\mathfrak{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is also a commutative algebra with multiplication $\mathfrak{x} \cdot \mathfrak{y} = \begin{bmatrix} x_1 y_1 \\ \vdots \\ x_n y_n \end{bmatrix}$ satisfying (1) for the derivative $\Xi = [\delta_j^i S]$. The following theorems can be obtained

THEOREM 1. *If n solving operations $X_1 x_1, \ldots, X_n x_n$ of the homogeneous system*

$$A_i^1 Sx_i + \sum_{j=1}^{n} A_{ij}^0 x_j = 0, \qquad i = 1, \ldots, n, \tag{2}$$

where A_i^1, A_{ij}^0 are endomorphisms of C^0, are multiplications, that is

$$X_i x_i = a_i x_i, \qquad i = 1, \ldots, n \tag{3}$$

then this system is fundamental if and only if the Wroński's determinant

$$|sa_1, sa_2, \ldots, sa_n|$$

is no divisor of zero.

THEOREM 2. *If solving operations* $X_0\, x_{00}, \ldots, X_{n-1}\, x_{n-1,0}$ *of the homogeneous equation*

$$A_n S^n x + A_{n-1} A^{n-1} x + \ldots + A_0 x = 0, \tag{4}$$

where $A_n, A_{n-1}, \ldots, A_0$ *are endomorphisms of* C^0, *are multiplications* (3), *then this system is fundamental if and only if the Wroński's determinant*

$$|sS^i a_j| \quad i = 0, 1, \ldots, n-1, \quad j = 1, \ldots, n$$

is no divisor of zero.

3. The method of variation of constants for equations with number coefficients

Let us consider a derivative equation

$$\alpha_n S^n x + \alpha_{n-1} S^{n-1} x + \ldots + \alpha_0 x = f, \quad \alpha_i \in \Gamma, \quad i = 1, \ldots, n, \quad \alpha_n \neq 0.$$

We suppose that characteristic equation

$$P(\lambda) = \alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \ldots + \alpha_0 = 0$$

has n roots $\lambda_1, \ldots, \lambda_n$ such that if $\lambda = \lambda_{k_1} + \ldots + \lambda_{k_m}$, $k_1, \ldots, k_m = 1, \ldots, n$, $m = 1, 2, \ldots$ then the equation

$$Sx - \lambda x = 0, \quad sx = 0$$

has only trivial solution $x = 0$. Then the results

$$\frac{x_0}{(I - \lambda_i T)^k} \cdot \frac{y_0}{(I - \lambda_j T)^l} = \frac{x_0\, y_0}{(I - (\lambda_i + \lambda_j) T)^{k+l}}$$

$i, j = 1, \ldots, n$, $k, l = 1, 2, \ldots$, and all their possible multiplications are well defined. We can extend the derivative S and the limit condition s on the space of results in such a way that the condition (1) is satisfied and the usual method of variation of constants can be applied if a given Wroński's determinamt W has the inverse. In particular, W has the inverse if the equations

$$Sx + (\lambda_1 + \ldots + \lambda_n) x = 0, \quad sx = 0$$

have only trivial solution, and the set $K(S)$ of constants is a field.

Example. The Wroński's determinants for ordinary differential and difference equations coincide with classic determinants in these theories, similarly as notions of fundamental systems. Similar determinants can be taken for the operators of Mikusiński.

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REFERENCES

[1] S. Bellert, *On foundations of operational calculus*, Bull. Acad. Polon. Sci., Cl. III, **5** (1957), 855—858.

[2] R. Bittner, *Operational calculus in linear spaces*, Studia Math., **20** (1961), 1—18.

[3] — , *On a new definition of polynomials*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **9** (1961), 79.

[4] — , *Algebraic properties of linear derivative equations in linear spaces*, Bull. Acad Polon. Sci., Sér. sci. math., astr. et phys., **9** (1961) [this issue].

[5] J. Mikusiński, *Operational calculus*, Warsaw, 1958.

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MATHÉMATIQUE

Sur la normale à l'hypersurface de l'espace affine A_n

par

T. HUSKOWSKI

Présenté par E. MARCZEWSKI le 3 janvier 1961

La normale affine à une surface de l'espace affine A_3 a été definie par G. Pick (voir [1], p. 116) dans le cas, où A_3 est basé sur le groupe unimodulaire, et par W. Slebodziński [2] dans le cas, ou A_3 est basé sur le groupe lineaire général. Dans la Note présente nous définirons la normale à une hypersurface de l'espace affine A_n à n dimensions basé sur le groupe général.

Soit R_n un repère arbitraire de l'espace A_n formé d'un point $M \in A_n$ et de n vecteurs I_j; ce repère dépend de n paramètres x^i (coordonnées du point M) et de n^2 paramètres v^i_j (coordonnées des vecteurs I_j)*). Le mouvement infinitésimal de R_n est défini par les équations

$$dM = \omega^i I_i, \quad dI_k = \omega^j_k \cdot I_j, \tag{1}$$

où les ω^i, ω^j_k sont les formes différentielles linéaires aux variables x^h, v^h_j, les n premières étant homogènes et indépendantes par rapport aux différentielles dx^h. Ces formes satisfont aux équations suivantes

$$d\omega^i = [\omega^k\, \omega^i_k], \quad d\omega^i_j = [\omega^i_k\, \omega^k_j], \tag{2}$$

qui expriment les conditions d'intégrabilité complète du système des équations (1).

Considérons maintenant une hypersurface $\mathfrak{M}_{n-1}$ de l'espace A_n donnée par les equations $x^h = f^h(u^1, u^2, \ldots, u^{n-1})$, où les f^h sont fonctions de classe C^3 définies dans un ouvert de l'espace numérique aux variables $u^\varkappa$. Attachons à un point arbitraire $M \in \mathfrak{M}_{n-1}$ un repère R_n choisi de manière que ses $n-1$ premiers vecteurs soient tangents aux lignes des coordonnées curvilignes $u^\varkappa$ dans $\mathfrak{M}_{n-1}$; il s'en suit que si R_n se déplace sur $\mathfrak{M}_{n-1}$, on a $\omega^n = 0$. Nous nous proposons de déterminer le vecteur I_n d'une manière intrinsèque, c'est-a-dire indépendante du choix des coordonnées x^i dans l'espace A_n et de celui des coordonnées curvilignes $u^\varkappa$ dans l'hypersurface $\mathfrak{M}_{n-1}$.

*) Dans toute ce qui suit les indices latins parcourent les valeurs $1, 2, \ldots, n$ et les indices grecs les valeurs $1, 2, \ldots, n-1$; l'indice qui se présente une fois en haut et une fois en bas dans un terme monôme remplace le signe de sommation.

Désignons à cet effet par Δ le déterminant composé des coordonnées des vecteurs I_j. Il est facile de vérifier qu'on a

$$\frac{d\Delta}{\Delta} = \omega_k^k. \tag{3}$$

Nous imposons au repère R_n attaché au point M les deux conditions

$$\omega_\alpha^n = \mu\omega_n^n, \tag{4}$$

$$[R\,\omega^n\,\omega_2^n \ldots \omega_{n-1}^n] = \varepsilon\,[\omega^1\,\omega^2 \ldots \omega^{n-1}], \quad \varepsilon = \pm 1, \tag{5}$$

où μ est une constante différente de zéro dont la valeur sera precisée plus loin. La condition (4) peut aussi prendre la forme

$$\omega_k^k = (\mu + 1)\,\omega_n^n \tag{6}$$

et, par suite, la formule (3) peut s'écrire de la manière suivante

$$\frac{d\Delta}{\Delta} = (\mu + 1)\,\omega_n^n. \tag{7}$$

Remarquons que les conditions (5) et (6) sont invariantes relativement aux transformations affines des coordonnées x^h de l'espace A_n.

Supposons maintenant que l'on introduit dans l'hypersurface $\mathfrak{M}_{n-1}$ de nouvelles coordonnées u^α au moyen d'une transformation de classe C^3; les vecteurs I_α se transformeront alors d'après les formules $\tilde{I}_\alpha = A_\alpha^\beta\, I_\beta, A_\alpha^\beta = \dfrac{\partial\bar{u}^\beta}{\partial u^\alpha}$. Pour obtenir un nouveau repère $\tilde{R}_n$ au lieu de R_n il faut adjoindre aux vecteurs $\tilde{I}_\alpha$ un vecteur $\tilde{I}_n$. Si l'on demande que la direction du vecteur I_n soit liée invariablement à l'hypersurface et que $\tilde{I}_n$ jouisse de la même propriété, il faut que $\tilde{I}_n = \lambda\, I_n$, où λ désigne une scalaire different de zéro. Le changement du repère R_n au point M est ainsi défini par les formules suivantes

$$\tilde{I}_\alpha = A_\alpha^\beta\, I_\beta, \quad \tilde{I}_n = \lambda\, I_n. \tag{8}$$

Il est facile de voir que les composantes du mouvement infinitésimal du repère $\tilde{R}_n$ sont données par les équations

$$\tilde{\omega}_\beta^\alpha = B_\beta^\alpha\omega^\beta, \quad \tilde{\omega}_\beta^n = \frac{1}{\lambda}\,A_\beta^\alpha\,\omega_\alpha^n, \quad \tilde{\omega}_n^n = \omega_n^n + \frac{d\lambda}{\lambda}, \tag{9}$$

où $\|B_\beta^\alpha\|$ désigne la matrice inverse de $\|A_\beta^\alpha\|$. Si l'on demande que la condition (5) soit satisfaite par les formes $\tilde{\omega}^\alpha$ et $\tilde{\omega}_n^\beta$, on obtient d'après les formules (9) la relation suivante

$$\frac{1}{\lambda^{n-1}}\,A[\omega_1^n \ldots \omega_{n-1}^n] = B[\omega^1 \ldots \omega^{n-1}],$$

$$A = |A_\beta^\alpha|, \quad B = |B_\beta^\alpha| = \frac{1}{A}.$$

En la comparant avec l'équation (5), on en déduit la relation

$$\lambda^{\frac{n-1}{2}} = A. \tag{10}$$

Reprenons maintenant la condition (4) et demandons qu'elle soit satisfaite par les formes $\tilde{\omega}$. En designant par $\tilde{\Delta}$ le déterminant des composantes des vecteurs $\tilde{I}_h$ on aura

$$\tilde{\Delta} = \lambda A \cdot \Delta, \quad \frac{d\tilde{\Delta}}{\tilde{\Delta}} = \frac{d\lambda}{\lambda} + \frac{dA}{A} + \frac{d\Delta}{\Delta}. \tag{11}$$

Les formes $\tilde{\omega}_h^h$ devant satisfaire à la relation $\frac{d\tilde{\Delta}}{\tilde{\Delta}} = (\mu + 1)\,\tilde{\omega}_n^n$ (cf. equ. (7)), la troisième des équations (9) et la deuxième des équations (11) entraînent la relation suivante

$$\frac{d\lambda}{\lambda} + \frac{dA}{A} + \frac{d\Delta}{\Delta} = (\mu+1)\left(\omega_n^n + \frac{d\lambda}{\lambda}\right).$$

Si l'on y tient compte de l'équation (7) on obtient

$$\frac{dA}{A} = \mu \frac{d\lambda}{\lambda},$$

d'où il résulte la relation

$$\lambda^n = cA. \qquad (c = \text{const}). \tag{12}$$

En la comparant avec la relation (10) on arrive aux égalités $c = 1$ et

$$\mu = \frac{n-1}{2}. \tag{13}$$

Par conséquent les deux conditions (4) et (5) deviennent

$$\omega_\alpha^\alpha = \frac{n-1}{2}\,\omega_n^n, \tag{14}$$

$$[\omega_1^n \dots \omega_{n-1}^n] = \varepsilon\,[\omega^1 \dots \omega^{n-1}], \qquad \varepsilon = \pm 1. \tag{15}$$

Nous allons montrer que les équations (14) et (15) déterminent complètement la normale affine à l'hypersurface $\mathfrak{M}_{n-1}$. Supposons dans ce but que les équations de $\mathfrak{M}_{n-1}$ soïent mises sous la forme $x^\alpha = u^\alpha$, $x^n = f(u^1, \dots, u^{n-1})$, f étant une fonction de classe C^3, et désignons par a^i les coordonnées du vecteur I_n et par f_α, $f_{\alpha\beta}$ les derivées $\frac{\partial f}{\partial u^\alpha}, \frac{\partial^2 f}{\partial u^\beta\,\partial u^\alpha}$; $L = |f_{\alpha\beta}|$. On aura alors $\omega^\alpha = du^\alpha$, $I_\alpha = \left\{\frac{\partial x^i}{\partial u^\alpha}\right\}$ et

$$\Delta = \left|\frac{\partial x^i}{\partial u^1}, \dots, \frac{\partial x^i}{\partial u^{n-1}}, a^i\right| = a^n - a^{n-1} f_1 - \dots - a^1 f_1.$$

Les équations $dI_i = \omega_i^k \cdot I_k$ deviennent maintenant

$$d\frac{\partial x^i}{\partial u^\alpha} = \omega_\alpha^\beta \frac{\partial x^i}{\partial u^\beta} + \omega_\alpha^n a^i,$$

$$da^i = \omega_n^\beta \frac{\partial x^i}{\partial u^\beta} + \omega_n^n a^i.$$

On en déduit les formules

$$\Delta \cdot \omega_\alpha^\alpha = -a^\alpha f_{\alpha\beta}\, du^\beta,$$
$$\Delta \cdot \omega_\alpha^n = f_{\alpha\beta}\, du^\beta.$$

La condition (14) prend alors la forme suivante

$$\frac{L}{\Delta^{n-1}} [du^1 \dots du^{n-1}] = \varepsilon [du^1 \dots du^{n-1}],$$

d'où il suit

$$\Delta^{n-1} = \varepsilon L, \tag{16}$$

ou le signe de ε doit être choisi de telle manière que cette équation puisse être resolue par rapport à Δ. On voit sur cette équation que la normale affine ne peut être définie que si $L \neq 0$. Dans le cas $n = 3$ l'inégalité $L \neq 0$ exclut seulement les points paraboliques de la surface. Nous supposerons donc $L \neq 0$. Comme d'après les formules (3) et (14) on a $\frac{n+1}{n-1}\omega_\alpha^\alpha = \frac{d\Delta}{\Delta}$, l'équation (15) peut être écrite comme suit

$$d\sqrt[n-1]{\varepsilon L} = \frac{n+1}{n-1} \Delta \cdot \omega_\alpha^\alpha = -\frac{n+1}{n-1} a^\alpha f_{\alpha\beta}\, du^\beta. \tag{17}$$

De cette équation et de l'équation (16) on trouve les relations

$$a^n - a^{n-1} f_{n-1} - \dots - a^1 f_1 = \sqrt[n-1]{\varepsilon L}, \quad f_{\alpha\beta}\, a^\alpha = -\frac{n-1}{n+1} \frac{\partial \sqrt[n-1]{\varepsilon L}}{\partial u^\beta} \tag{18}$$

dont les $n-1$ dernières forment un système d'équations linéaires au déterminant $L \neq 0$; elles permettent de trouver les inconnues a^α, la première équation déterminant ensuite l'inconnue a^n ce qui démontre notre proposition.

La notion de la normale affine étant ainsi établie nous pouvons définir sur l'hypersurface $\mathfrak{M}_{n-1}$ une connexion induite au sens de Levi-Civita, dont les équations de structure ont la forme suivante

$$d\omega^\alpha = [\omega^\beta \omega_\beta^\alpha], \qquad d\omega_\beta^\alpha = [\omega_\beta^\gamma \omega_\gamma^\alpha] + R_{\sigma\varkappa\beta}{}^\alpha [\omega^\sigma \omega^\varkappa],$$

les $R_{\sigma\varkappa\beta}{}^\alpha$ étant les composantes de la courbure. Il résulte des équations (3) et (14) que l'on a $d\omega_\alpha^\alpha = 0$ ou $R_{\sigma\varkappa\alpha}{}^\alpha = 0$ ce qui montre que la connexion induite admet une unité absolue de volume.

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OUVRAGES CITÉS

[1] W. Blaschke, *Vorlesungen über Differentialgeometrie II*, Berlin, 1923.

[2] W. Ślebodziński, *Sur quelques problèmes de la théorie des surfaces de l'espace affine*, Prace mat.-fiz. **46** (1938), 291—346.

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MATHÉMATIQUE

Sur l'application géodésique d'une variété riemannienne sur l'espace récurrent

par

W. ROTER

Présenté par E. MARCZEWSKI le 3 janvier 1961

Soient V^n et $\bar{V}^n$ deux variétés riemanniennes, à n dimensions de métrique définie positive; désignons par Γ^k_{ij} ($\bar{\Gamma}^k_{ij}$) les coefficients de la structure, par R^h_{ijk} ($\bar{R}^h_{ijk}$) — les composantes de la courbure de V^n ($\bar{V}^n$) et par ∇_l ($\bar{\nabla}_l$) — la différentiation covariante dans V^n ($\bar{V}^n$).

On sait que V^n et $\bar{V}^n$ sont applicables géodésiquement l'une sur l'autre, si entre leurs points une correspondance peut être établie telle qu'il soit

$$\Gamma^k_{ij} = \bar{\Gamma}^k_{ij} + \delta^k_i \psi_j + \delta^k_j \bar{\psi}_i, \tag{1}$$

$$R^h_{ijk} = \bar{R}^h_{ijk} + \delta^h_k \bar{\psi}_{ij} - \delta^h_j \psi_{ik}, \tag{2}$$

où $\bar{\psi}_i$ désigne un vecteur covariant et $\psi_{jk} = \nabla_j \psi_i - \psi_i \bar{\psi}_j$; dans la suite en parlant d'une application géodésique nous admettrons constamment $\psi_i \neq 0$. Nous consacrerons cette Note à l'étude du cas, où V^n est un espace récurrent introduit par H. S. Ruse [1] et défini par la relation

$$\nabla_l R_{hijk} = \varphi_l R_{hijk}, \tag{3}$$

φ_l étant un gradient différent de zéro; cet espace a été désigné par $\mathfrak{R}_n$.

I. S. Sinioucov a montré [2] que si V^n est un espace de E. Cartan ($\nabla_l R_{hijk} = 0$), la variété $\bar{V}^n$ est un espace à courbure constante: nous allons généraliser le résultat de Sinioucov en démontrant le suivant

THÉORÈME. *Une variété riemannienne applicable géodésiquement sur l'espace récurrent est un espace à courbure constante.*

Pour prouver ce théorème supposons que la variété V^n est un espace récurrent. La différentiation covariante de l'équation (3), suivie d'une alternation des indices, conduit à la relation

$$\nabla_m \nabla_l R_{hijk} - \nabla_l \nabla_m R_{hijk} = (\nabla_m \varphi_l - \nabla_l \varphi_m) R_{hijk}.$$

Le vecteur φ_l étant un gradient, cette relation peut être ramenée à la forme suivante

$$\nabla_m \nabla_l R_{hijk} - \nabla_l \nabla_m R_{hijk} = 0.$$

Si l'on y applique la formule de commutation des dérivées covariantes, il vient

$$R^h_{.rjk}\,R^r_{.ilm}+R^h_{.irk}\,R^r_{.jlm}+R^h_{.ijr}\,R^r_{.klm}-R^r_{.ijk}\,R^h_{.rlm}=0. \tag{4}$$

Remarquons encore qu'en contractant les indices dans les équations (2) et (3) on obtient (respectivement) les deux relations suivantes

$$R_{ij}=\bar{R}_{ij}+(n-1)\,\psi_{ij}, \tag{5}$$

$$\nabla_l R_{ij}=\varphi_l R_{ij}, \tag{6}$$

où R_{ij} est le tenseur de Ricci. Les conditions d'intégrabilité des équations (6) entraînent ensuite l'équation suivante

$$R_{hj}\,R^h_{.ikl}+R_{ih}\,R^h_{.jkl}=0. \tag{7}$$

Or, remarquons que des équations (2), (4), (5) et (7), qui sont valables aussi pour l'espace de Cartan, Sinioucov a conclu qu'il doit être

a) $$R_{ij}=\frac{R}{n}\,g_{ij}$$

ou bien

b) $$\Delta\bar{g}_{ij}-n\bar{\psi}_{ij}=0,$$

g_{ij} étant le tenseur métrique de V^n et $\Delta=\bar{\psi}_{ij}\,\bar{g}^{ij}$.

Nous allons considérer séparément ces deux cas.

a) Observons d'abord que des équations (2) et (3) resulte la relation

$$W^h_{.ijk}=\bar{W}^h_{.ijk} \tag{8}$$

où W^h_{ijk} ($\bar{W}^h_{ijk}$) est le tenseur de Weyl de la variété $\mathfrak{R}_n$ ($\bar{V}^n$). Le tenseur de Weyl étant défini par la formule

$$W^h_{.ijk}=R^h_{.ijk}-\frac{1}{n-1}\,(\delta^h_k\,R_{ij}-\delta^h_j\,R_{ik}),$$

on en déduit, en tenant compte de l'équation (3), la relation suivante

$$\nabla_l W^h_{.ijk}=\varphi_l W^h_{.ijk}$$

et par suite, selon (8),

$$\nabla_l \bar{W}^h_{.ijk}=\varphi_l \bar{W}^h_{.ijk}.$$

En vertu de l'équation (3) la dernière relation peut être écrite de la manière suivante

$$\bar{\nabla}_l \bar{W}^h_{.ijk}+\delta^h_l\,\bar{\psi}_r\,\bar{W}^r_{.ijk}-2\bar{\psi}_l\bar{W}^h_{.ijk}-\bar{\psi}_i\,\bar{W}^h_{.ljk}- \tag{9}$$

$$-\bar{\psi}_j\,\bar{W}^h_{.ilk}-\bar{\psi}_k\,\bar{W}^h_{.ijl}=\varphi_l\,\bar{W}^h_{.ijk}.$$

Dans le cas a) le tenseur W_{hijk} est antisymétrique par rapport à ses deux premiers indices; donc, si l'on abaisse l'indice h dans l'équation (9) et que l'on calcule sa partie symétrique par rapport aux indices h et i, on trouve

$$\bar{g}_{hl}\,\bar{\psi}_r\,\bar{W}^r_{.ijk}+\bar{g}_{il}\,\bar{\psi}_r\,\bar{W}^r_{.hjk}-\bar{\psi}_i\,\bar{W}_{hljk}-\bar{\psi}_h\,\bar{W}_{iljk}=0$$

d'où il résulte, comme l'a montré Sinioucov, que la variété est à courbure constante: c'est ce que nous avions à montrer.

b) Pour démontrer le théorème dans le cas b) posons

$$\bar{\Pi}^h_{.ijk} = \bar{R}^h_{.ijk} + \frac{\Delta}{n}(\delta^h_k \bar{g}_{ij} - \delta^h_j \bar{g}_{ik}).$$

Comme on a

$$\bar{\Pi}^h_{.ijk} = R^h_{.ijk},$$

il en résultera la relation suivante

$$\nabla_l \bar{\Pi}^h_{.ijk} = \nabla_l R^h_{.ijk} = \varphi_l R^h_{.ijk}.$$

et par suite

$$\bar{\nabla}_l \bar{\Pi}^h_{.ijk} = \varphi_l \Pi^h_{.ijk}.$$

Un raisonnement tout pareil à celui du cas a) démontre la thèse du Théorème.

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OUVRAGES CITÉS

[1] H. S. Ruse, *Three-dimensional spaces of recurrent curvature*, Proc. London Math. Soc. **50** (1949), 438—446.

[2] I. S. Sinioucov, *Sur l'application géodésique des variétés riemanniennes sur un espace riemannien symétrique* [en russe], DAN RUSS, **98** (1954), 21—23.

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MATHÉMATIQUE

Systèmes de commande et équations au contingent

par

T. WAŻEWSKI

Présenté le 31 décembre 1960

A. Marchaud et S. K. Zaremba ont introduit et développé les théories équivalentes entre elles (cf. [3]) des „équations" au contingent et au paratingent [2], [3]. Dans la présente note nous introduisons la notion de l'équation au contingent S^* „associée" au système de commande donné S (cf. par. **4**) et indiquons les conditions dans lesquelles S et S^* sont équivalents entre eux. Il devient ainsi possible d'appliquer au système de commande S certains résultats de ces auteurs relatifs à S^*. On obtient certains théorèmes sur l'existence des solutions „periphériques" (cf. par. **8**) de S qui réalisent le temps minimum de rencontre d'un but fixe ou mobile (Théorèmes 2 et 3). Nous traitons le cas non linéaire et indiquons ensuite une application au cas où S est linéaire relativement aux variables de commande u, mais non linéaire par rapport aux variables de position x. Le Théorème 1, dont les autres résultent facilement, est présenté sans démonstration. Celle-ci paraîtra dans un autre article.

1. Notations et définitions préliminaires. I. R_q désigne l'espace euclidien réel à q dimensions. La relation

$$A \in \mathrm{comp}\,(R_q) \tag{1.1}$$

exprimera que

$$A \subset R_q, \quad A \text{ est compact et non vide.} \tag{1.2}$$

Pour un tel ensemble A, le symbole

$$V(A, r) \tag{1.3}$$

désigne le voisinage fermé de A de rayon $r > 0$.

II. Soit w un point variable dans R_q. Soit $B \subset R_q$ et soit $N(w) \subset R_q$ un ensemble dépendant de $w \in B$. On dit que $N(w)$ est *semicontinue par inclusion dans* B et on écrit

$$N(w) \in \text{sem. incl. } (B) \tag{1.4}$$

lorsque, pour chaque $w_0 \in B$ et $m > 0$, il existe un $s > 0$ tel que $N(w) \subset V(N(w_0), m)$ et lorsque $w \in B$. $V(w_0, s)$.

III. Soit t le temps, J un intervalle et $x(t) = (x_1(t), \ldots, x_n(t))$ une courbe définie pour $t \in J$. Posons $p(t) = (x(t) - x(t_0))/(t - t_0)$. L'ensemble $M \subset R_n$ sera dit *dérivée contingentielle* de $x = x(t)$ à l'instant $t_0 \in J$ lorsque 1°) les relations

$$t_n \in J, \quad t_n \neq t_0, \quad t_n \to t_0, \quad p(t_n) \to p_0 \tag{1.5}$$

entraînent la relation $p_0 \in M$, 2°) à chaque $p_0 \in M$ correspond une suite t_n vérifiant (1.5). La dérivée contingentielle de $x(t)$ à l'instant t sera désignée par

$$D^* x(t). \tag{1.6}$$

IV. La fonction $u \quad u(t) = (u_1(t), \ldots, u_p(t))$ sera dite *mesurable* dans l'intervalle J lorsque 1°) elle est définie presque partout dans J, c.à.d. pour $t \in J - Z$, où Z est un ensemble de mesure nulle, 2°) les fonctions composantes $u_1(t), \ldots, u_p(t)$ de $u(t)$ sont mesurables au sens de Lebesgue dans $J - Z$.

2. Système de commande. Nous introduisons les variables réelles $x \in R_n$, $u \in R_p$, t = temps et la fonction vectorielle aux valeurs situées dans R_n : $f(t, x, u) = (f_1(t, x, u), \ldots, f_n(t, x, u))$ définie dans un ensemble $A \subset R_1 \times R_n \times R_p$. Nous introduisons ensuite un ensemble non vide $C(t, x) \subset R_p$ dépendant de (t, x) où (t, x) varie dans un ensemble fixe $B \subset R_1 \times R_n$. Nous supposons que $(t, x, u) \in A$ lorsque $(t, x) \in B$, $u \in C(t, x)$. Soit $J = J(b) = [0, b]$ un intervalle. Envisageons le système (S) des conditions suivantes

$$(S) \quad \begin{cases} x(t) = \int_0^t f(s, x(s), u(s))\, ds \text{ pour } t \in J, \\ u(t) \text{ } H\text{est mesurable dans } J, \\ u(t) \in C(t, x(t)) \text{ presque partout dans } J. \end{cases}$$

Le système des conditions (S), dans lequel J, $x(t)$, $u(t)$ sont considérés comme inconnus, sera dit *système de commande* avec la condition initiale

$$x(0) = O_n, \tag{2.1}$$

où O_n désigne l'origine des coordonnées de R_n. La fonction $u(t)$ sera dite *fonction de commande*, l'ensemble $C(t, x)$ — domaine de commande. Toute fonction $x(t)$ pour laquelle existent $u(t)$ et J, tels que les conditions (S) sont satisfaites, sera dite *caractéristique* issue de O_n (dans la suite tout court caractéristique) de (S).

3. Contredomaine de commande. Posons $F_{t,x}(u) = f(t, x, u)$. La formule

$$v = F_{t,x}(u) \tag{3.1}$$

représente une famille de transformations dépendant des paramètres t, x. L'ensemble

$$E(t, x) = F_{t,x}(C(t, x)) \tag{3.2}$$

constituant l'image de l'ensemble $C(t, x)$ par l'intermédiaire de la transformation (3.1), sera dit contredomaine de commande relatif à (S).

4. Équation au contingent (S^*) associée au système de commande (S). Le système des deux conditions (cf. (1.6), (3.2), (2.1))

$$(S^*) \qquad D^* x(t)\, \varepsilon \in (t, x) \text{ partout dans } J$$

et $x(0) = O_n$ sera dite „équation" au contingent associée au système de commande (S). Dans (S^*) la fonction $x(t)$ et J sont considérés comme inconnus. Chaque fonction $x(t)$ vérifiant (S^*) sera dite *caractéristique* issue de l'origine (ou, tout court, caractéristique) de (S^*).

5. Equivalence de (S) et (S^*). Si chaque caractéristique de (S) est une caractéristique de (S^*) et inversement, alors (S) et (S^*) seront dits équivalents.

6. Hypothèse H_1. *I. Nous introduisons les variables $x \in R_n$, $u \in R_p$, t = temps. Chacun des trois cas $n = p$, $n > p$, $p > n$ est admis.*

II. La boule fermée à n dimensions de centre O_n (cf. (2.1)) et de rayon r est désignée par $B(r)$.

III. $J(a) = [0, a]$ est un intervalle compact $(a > 0)$.

IV. K est un ensemble fixe et (cf. (1.1)) $K \in \text{comp}\ (R_p)$.

V. La fonction $f(t, x, u) = (f_1(t, x, u), \ldots, f_n(t, x, u))$ est continue dans un ensemble ouvert contenant le produit cartésien $W = J(a) \times B(r) \times K$.

VI. Pour un $M > 0$ on a $f(t, x, u) \in B(M)$ lorsque $(t, x, u) \in W$.

VII. $C(t, x)$ est un ensemble dépendant de t, x. Pour $(t, x) \in J(a) \times B(r)$ on a (cf. (1.1)) $C(t, x) \subset K$, $C(t, x) \in \text{comp}\ (R_p)$.

VIII. $C(t, x) \in$ sém. incl $(J(a) \times B(r))$ (cf. (1.4)).

IX. h = minimum $(a, r/M)$.

Hypothèse H_2. *L'ensemble $E(t, x)$ (cf. (3.2) est convexe pour $(t, x) \in J(a) \times B(r)$ (l'ensemble $E(t, x)$ étant convexe, il est, à plus forte raison, connexe).*

Théorème 1. *Si les Hypothèses H_1 et H_2 sont admises toutes les deux, alors le système de commande (S) et l'équation au contingent (S^*) qui lui est associée sont équivalents* (cf. par. **5**).

7. Convention 1. Au cas où les Hypothèses H_1 et H_2 sont vérifiées nous conviendrons d'appeler, tout court, caractéristiques: 1°) les caractéristiques (issues de l'origine O_n) de (S) et 2°) celles de (S^*).

Cette convention est justifiée par le Théorème 1.

8. Définition 1. Admettons les Hypothèses H_1 et H_2.

I. L'ensemble N des points (t, x) qui est engendré par toutes les caractéristiques (cf. par. 7) sera dit *zone d'émission* du point $t = 0$, $x = O_n$ (cf. (2.1)).

II. Soit Q^* la frontière de N et Q_1 la partie de Q^* composée des points (t, x) pour lesquels t est différent de h. La fermeture de Q_1 sera dite *frontière latérale* de N. Elle sera désignée par

$$(8.1) \qquad Q = \text{front. lat.}\ (N).$$

III. La caractéristique $x = x(t)$ sera dite *périphérique* dans l'intervalle $[0, t^*]$ lorsque (cf. (8.1)) $(t, x(t)) \in$ front. lat. (N) pour $t \in [0, t^*]$.

IV. Soit $A(t) \subset R_n$ un ensemble mobile pour $t \in J(h)$. On dira que $A(t)$ est *accessible* (par les caractéristiques — cf. par. 7) lorsque pour $t_0 \in J(h)$ l'ensemble $N(t_0) \cdot A(t_0)$ n'est pas vide. Si la classe de tels t_0 admet un minimum t^*, alors t^* sera *temps minimum de rencontre* de $A(t)$ (par les caractéristiques). Toute caractéristique $x = x(t)$ pour laquelle $(t^*, x(t^*)) \in (t^* \times A(t^*))$ sera dite caractéristique optimale de rencontre de $A(t)$. Si $u(t) \in$ frontière de $C(t, x(t))$ presque partout, alors $u(t)$ sera dite périphérique.

9. Théorème 2. *Admettons les Hypothèses* H_1 *et* H_2. *Ceci admis, nous avons*:

I. Chaque caractéristique (cf. par. 7) *peut être prolongée sur l'intervalle* $J(h)$.

II. La zone d'émission N (cf. par. 8) *ainsi que ses sections par les plans* $t = t_0$ *(où* $t_0 \in J(h)$*) sont connexes et compactes.*

III. Chaque point (cf. par. **8**, II) $P^\circ = (t^\circ, x^\circ) \in$ *front. lat.* (N) *peut être joint à l'origine par une caractéristique* $x = x(t)$ *qui est périphérique dans l'intervalle* $[0, t^\circ]$ (cf. par. 8, III).

IV. Soit $A(t) \in \mathrm{comp}(R_n)$ (cf. (1.1)) *un ensemble dont la position varie d'une façon continue au sens de Hausdorff* (cf. [1], p. 293) *pour* $t \in J(h)$. *Supposons que l'origine* O_n (cf. (2.1)) *n'appartient pas à* $A(0)$. *Si* $A(t)$ *est accessible par les caractéristiques* (cf. par. **8**, IV) *alors le temps optimal* t^* *de rencontre de* $A(t)$ *par les caractéristiques existe et une au moins des caractéristiques optimales est périphérique* (cf. par. **8**, IV, III).

Démonstration. Les parties I, II, III de la thèse du théorème ci-dessus se trouvent établies dans [3] relativement à l'équation au contingent (S^*).

La partie IV en résulte facilement. En vertu du Théorème 1, notre Théorème 2 reste vrai relativement au système de commande (S).

Théorème 3. *Admettons les Hypothèses* H_1 *et* H_2. *Supposons que* $n = p$ (cf. Hyp. H_1, I). *Admettons que toute transformation de la famille* (3.1) *soit un homéomorphisme.*

Ceci étant admis, on trouve qu' à toute caractéristique périphérique (cf. par. **8**, III) *correspond une fonction de commande* $u = u(t)$ *périphérique, c.à.d. telle que, pour presque tous les* $t \in J(h)$, *le point* $u(t)$ *appartient à la frontière de l'ensemble* $C(t, x(t))$. *Si l'ensemble* $A(t)$ *(jouissant des properiétés indiquées dans le Théorème* 2*) est accessible* (cf. par. **8**, **IV**) *alors les caractéristiques optimales périphériques de rencontre de* $A(t)$ *existent et chacune d'elles correspond à une fonction de commande périphérique.*

Démonstration. Théorème 3 résulte du Théorème 1 car les transformations (3.1) transforment la frontière de $E(t, x)$ en frontière de $C(t, x)$.

Théorème 4. *Admettons l'Hypothèse* H_1 *et admettons que la fonction* $f(t, x, u)$ *est linéaire en* u *dans l'ensemble* W (cf. Hyp. H_1, V) *ou, plus généralement, que la transformation* (3.1) *soit projective en* u *dans* $C(t, x)$ *(pour* $(t, x) \in J(a) \times B(r)$*).*

Si nous admettons en plus que les domaines de commande $C(t, x)$ *soient convexes alors les contredomaines de commande* $E(t, x)$ *sont convexes elles aussi, et l'Hypothèse* H_2 *est remplie.*

Dans ce cas on a donc la possibilité d'appliquer les Théorèmes 1, 2, 3 car les prémisses de ces théorèmes se trouvent vérifiées.

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OUVRAGES CITÉS

[1] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914.

[2] A. Marchaud, *Sur les champs de demi-cônes et les équations différentielles du premier ordre*, Bull. de la Soc. Math. de France, **62** (1934).

[3] S. K. Zaremba, *Sur les équations au paratingent*, Bull. des Sci. Math., **60** (1936).

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MATHEMATICS

A Note on Modular Spaces. I

by

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1. Let X denote a real linear space, ϱ — a functional defined in X, $-\infty < \varrho(x) \leqslant \infty$. The functional ϱ will be called a *modular*, if the conditions

A. 1. $\varrho(0) = 0$; A. 2. $\varrho(\lambda x) = 0$ for all λ implies $x = 0$; A. 3. $\varrho(-x) = \varrho(x)$;

A. 4. $\varrho(\alpha x + \beta y) \leqslant \varrho(x) + \varrho(y)$ for $x, y \in X$, $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$.

1.1. A functional ϱ will be called *s-convex*, $0 < s \leqslant 1$, if

A_s. 4. $\varrho(\alpha x + \beta y) \leqslant \alpha^s \varrho(x) + \beta^s \varrho(y)$ for $x, y \in X$, $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$.

A 1-convex functional will be called briefly *convex*.

Evidently, A_1. 4. implies A. 4. If $\varrho(0) = 0$ for an s-convex functional ϱ, then $\varrho(ax) \leqslant a^s \varrho(x)$ for $0 \leqslant a \leqslant 1$ and consequently, $\varrho(tx)\, t^{-s}$ is a nondecreasing function of t, $t > 0$.

1.2. The following properties of modulars are of importance in our considerations:

B. 1. $a_n \to 0$ implies $\varrho(a_n x) \to 0$;

B. 2. $\varrho(x_n) \to 0$ implies $\varrho(\alpha x_n) \to 0$ for every $\alpha > 0$.

1.21. We shall apply the following notation: $X_\varrho^* = \{x : \varrho(x) < \infty,\ x \in X\}$, $X^*(\varrho) = \{x : \varrho(\lambda x) < \infty$ for a certain $\lambda(x) > 0,\ x \in X\}$. Of course, every x satisfying B. 1. belongs to $X^*(\varrho)$. Denote by $X(\varrho)$ the set of x satisfying B. 1. $X^*(\varrho)$ is a linear space, $X(\varrho)$ — its subspace.

1.3. It is shown in [2] that in $X(\varrho)$ a norm may be defined as follows *):

$$(*) \qquad \|x\|_\varrho = \inf\{\varepsilon > 0 : \varrho(x/\varepsilon) \leqslant \varepsilon\},$$

possessing the following properties: a) $\|\ \|_\varrho$ is an F-norm, b) $\varrho(x) \leqslant \|x\|_\varrho$ for $\|x\|_\varrho < 1$, c) $\|\lambda x\|_\varrho$ is nondecreasing for $\lambda \geqslant 0$. In the sequel, we call this norm

*) In [2] the norm (*) is defined assuming a condition stronger that A. 2. However, the proof that (*) is a norm possessing the properties in 1.3 can be performed by our assumption A.2 without any changes. In [3] modulars, defined as in 1, were termed semimodulars.

the norm generated by the modular ϱ. Under some additional assumptions on the modular ϱ, other definitions of the norm may be introduced in $X(\varrho)$ by means of ϱ. There holds the theorem:

1.4. *If* ϱ *is an s-convex modular, then* $X(\varrho) = X^*(\varrho)$ *and an s-homogeneous norm can be defined in* $X^*(\varrho)$ *as follows*:

$$(+) \qquad \|x\|_{0s} = \inf\left\{\varepsilon > 0 : \varrho\left(\frac{x}{\varepsilon^{1/s}}\right) \leqslant 1\right\}.$$

We prove, e.g., the triangle inequality. Let $\|x\|_{0s} < \varepsilon_1$, $\|y\|_{0s} < \varepsilon_2$, $\varepsilon = \varepsilon_1 + \varepsilon_2$. Then

$$\varrho\left((x+y)\,\varepsilon^{-1/s}\right) = \varrho\left(\varepsilon_1^{1/s}\,\varepsilon^{-1/s}\,x\,\varepsilon_1^{-1/s} + \varepsilon_2^{1/s}\,\varepsilon^{-1/s}\,y\,\varepsilon_2^{-1/s}\right) \leqslant$$

$$\leqslant (\varepsilon_1^{1/s}\,\varepsilon^{-1/s})^s\,\varrho(x\,\varepsilon_1^{-1/s}) + (\varepsilon_2^{1/s}\,\varepsilon^{-1/s})^s\,\varrho(y\,\varepsilon_2^{-1/s}) \leqslant 1,$$

for $(\varepsilon_1^{1/s}\,\varepsilon^{-1/s})^s + (\varepsilon_2^{1/s}\,\varepsilon^{-1/s})^s = 1$. Thus, $\|x+y\|_{0s} \leqslant \varepsilon_1 + \varepsilon_2$.

1.41. The norms $\|\ \|_{0s}$ and $\|\ \|_\varrho$ are equivalent, more exactly, there hold the inequalities: (a) $\|x\|_{0s} \leqslant (\|x\|_\varrho)^s$, if $\|x\|_\varrho < 1$, (b) $(\|x\|_\varrho)^{s+1} \leqslant \|x\|_{0s}$, if $\|x\|_{0s} < 1$.

If $\|x\|_\varrho < 1$ and $\|x\|_\varrho < \varepsilon < 1$, then $\varrho(x\,(\varepsilon^s)^{-1/s}) \leqslant \varepsilon < 1$, whence $\|x\|_{0s} \leqslant \varepsilon^s$. If $\|x\|_{0s} < 1$ and $\|x\|_{0s} < \varepsilon < 1$, then $\varrho(x\,\varepsilon^{-1/1+s}) \leqslant (\varepsilon^{1/s}\ \varrho(x\varepsilon^{-1/1+s}) \leqslant \varepsilon^{1/1+s}$, whence $\|x\|_\varrho \leqslant \varepsilon^{1/1+s}$.

1.5. If ϱ is an s-convex modular, then

$$\|x\|_{0s} = \inf_{t>0}[\sup(t^{-s},\ \varrho(tx)\,t^{-s})].$$

It follows from (+) that $\varrho(x) \leqslant 1$ implies $\|x\|_{0s} \leqslant 1$. If $\varrho(x) > 1$, then $\varrho(x\varrho(x)^{-1/s}) \leqslant (\varrho(x)^{-1/s})^s\,\varrho(x) = 1$ and $\|x\varrho(x)^{-1/s}\|_{0s} = \|x\|_{0s}\,\varrho(x)^{-1} < 1$. Hence there follow the inequalities

$$\|x\|_{0s} \leqslant \sup(1,\ \varrho(x)),\ \|x\|_{0s} \leqslant \sup(t^{-s},\ \varrho(tx)\,t^{-s}) \text{ for } t > 0,$$

i.e. $\|x\|_{0s} \leqslant \inf_{t>0}[\sup(t^{-s},\ \varrho(tx)\,t^{-s}]$. The definition of the norm (+) implies that if $0 < t^0 < (\|x\|_{0s})^{-1/s}$, then $\varrho(t_0 x) \leqslant 1$, i.e.

$$\inf_{t>0}[\sup(t^{-s},\ \varrho(tx)\,t^{-s}] \leqslant \sup(t_0^{-s},\ \varrho(t_0 x)\,t_0^{-s}) \leqslant t_0^{-s},$$

where t_0^{-s} is arbitrarily near to $\|x\|_{0s}$.

1.51. If ϱ is an s-convex modular, then the functional

$$\|x\|_s = \inf_{t>0}(t^{-s} + \varrho(tx)\,t^{-s})$$

is an s-homogeneous norm in $X^*(\varrho)$ and there hold the inequalities

$$(\circ) \qquad \tfrac{1}{2}\,\|x\|_s \leqslant \|x\|_{0s} \leqslant \|x\|_s.$$

Denoting $\gamma(x) = 1 + \varrho(x)$, we have $\|x\|_s = \inf_{t>0} \gamma(tx)\, t^{-s}$. Moreover, $0 \leqslant \leqslant \|x\|_s < \infty$; in fact, $\varrho(x) \geqslant 0$ for $x \in X$ and $\varrho(tx) < \infty$ for $x \in X^*(\varrho)$, if $t > 0$ is sufficiently small. The s-homogeneity being obvious, we prove only the triangle inequality. Let $t', t'' > 0$; $\gamma(x)$ is an s-homogeneous functional, whence

$$\gamma\left(\frac{t'^{1/s}\, t''^{1/s}}{(t'+t'')^{1/s}}(x+y)\right) = \gamma\left(\alpha t'^{1/s} x + \beta t''^{1/s} y\right) \leqslant \alpha^s \gamma(t'^{1/s} x) + \beta^s \gamma(t''^{1/s} y),$$

where

$$\alpha = t''^{1/s}(t'+t'')^{-1/s}, \qquad \beta = t'^{1/s}(t'+t'')^{-1/s}, \qquad \alpha^s + \beta^s = 1.$$

Choosing $t_0 = t'^{1/s}\, t''^{1/s}(t'+t'')^{-1/s}$ we obtain

$$\|x+y\|_s \leqslant \gamma(t_0(x+y))\, t_0^{-s} \leqslant \gamma(t'^{1/s}x)\, t'^{-1} + \gamma(t''^{1/s} y)\, t''^{-1}.$$

The inequalities (o) follow from **1.5**, $\frac{1}{2}\gamma(tx)\, t^{-s} \leqslant \sup(t^{-s}, \varrho(tx)\, t^{-s}) \leqslant \leqslant \gamma(tx)\, t^{-s}$. If $\|x\|_s = 0$, then (o) implies $\|x\|_{0s} = 0$, i.e. $x = 0$. Hence, $\|\ \|_s$ is a norm and not only a quasinorm.

1.6. If ϱ is an s-convex modular and if $n(x)$ is a quasinorm in $X^*(\varrho)$ continuous with respect to the norm $\|\ \|_\rho$, then

(a) $$\sup_{\varrho(x) \leqslant 1} n(x) = k$$

is finite; moreover, if $n(x)$ is s-homogeneous, then

(b) $$n(x) \leqslant k(1 + \varrho(x)).$$

We prove (a). Suppose $\|x\|_\rho \leqslant \eta$ implies $n(x) \leqslant 1$. If x satisfies the inequality $\varrho(x) \leqslant 1$, we choose a positive integer l such that $l\eta > 1$, $l > \eta^{-(s+1)/s}$. We have $\varrho(l^{-1}\eta^{-1}x) \leqslant l^{-s}\eta^{-s} < \eta$, i.e. $\|l^{-1}x\|_\rho \leqslant \eta$, thence $n(l^{-1}x) \leqslant 1$, $n(x) \leqslant l$. Now, we prove (b). If $\varrho(x) > 1$, then $\varrho(x\,\varrho(x)^{-1/s}) \leqslant 1$, whence $n(x\,\varrho(x)^{-1/s}) \leqslant k$, $n(x)\,\varrho(x)^{-1} \leqslant k$; thus $n(x) \leqslant k \sup(1, \varrho(x))$ for an arbitrary $x \in X^*(\varrho)$.

The last inequality implies easily that, by the assumptions of **1.6** (b), $n(x) \leqslant \leqslant k\|x\|_{0s}$.

1.61. If ϱ is an arbitrary modular and if $n(x)$ is an s-homogeneous quasinorm in $X(\varrho)$ satisfying the inequality

$$n(x) \leqslant k(1 + \varrho(x)) \quad \text{for} \quad x \in X(\varrho),$$

then $\|x_n\|_\rho \to 0$ implies $n(x_n) \to 0$.

Evidently, $n(x) \leqslant k(t^{-s} + \varrho(tx)\, t^{-s})$, $t > 0$; it suffices to note that $\|x_n\|_\rho \to 0$ implies $\varrho(tx_n) \to 0$ for an arbitrary $t > 0$.

1.7. If ϱ is a convex (i.e. a 1-convex) modular, then the norm $\|\ \|_1$ is the Amemiya's norm, well-known in the theory of modular spaces (cf. [4]). It can be also defined as follows. By the definition of the norm $\|\ \|_1$, a distributive functional ξ over $X^*(\varrho)$ is continuous with respect to the norm $\|\ \|_1$ and has a norm $\|\xi\| \leqslant 1$ if and only if

(++) $$\xi(x) \leqslant 1 + \varrho(x) \text{ for } x \in X^*(\varrho).$$

Let $n(x) = \sup \xi(x)$, where the supremum is taken over all distributive functionals over $X^*(\varrho)$ satisfying $(++)$. Since for every ξ satisfying $(-+)$ we have $\xi(x) \leqslant t^{-1} - \varrho(tx)\, t^{-1}$ for $t > 0$, there holds $n(x) \leqslant \|x\|_1$. On the other hand, there exists a distributive functional ξ continuous in $X^*(\varrho)$ with respect to $\| \ \|_1$ satisfying the conditions $\xi(x_0) = \|x_0\|_1$, $\xi(x) \leqslant \|x\|_1 \leqslant 1 + \varrho(x)$; hence, $n(x_0) \geqslant \|x_0\|_1$. Thus, $n(x) = \|x\|_1$.

1.8. A modular ϱ satisfies the *conditions* (D), if it has the following property: to every $0 < \alpha < 1$ there exists a decomposition $x = x_1 + x_2$ such that $\varrho(x_1) \leqslant \alpha\varrho(x)$, $\varrho(x_2) \leqslant (1 - \alpha)\varrho(x)$.

Convex modulars are examples of modulars satisfying the condition (D). In **2** we give further examples of such modulars.

1.81. If ϱ is a modular satisfying the condition (D) and if $n(x)$ is a quasinorm in $X(\varrho)$ continuous with respect to $\| \ \|_\varrho$, then **1.6** (a), (b) hold, where $k < \infty$.

Let $\varrho(x) \leqslant 1$. Given any positive integer k, there exists a decomposition $x = x_1 + x_2 + \ldots + x_{2k}$, $\varrho(x_i) \leqslant 2^{-k}\varrho(x) \leqslant 2^{-k}$. Assume $\|x\|_\varrho < l^{-1}$ (l — an integer) implies $n(x) \leqslant 1$; choosing k sufficiently large (k is independent of x) we have $\varrho(lx_i\, l^{-1}) < l^{-1}$, whence $\|l^{-1} x_i\|_\varrho < l^{-1}$, $n(l^{-1} x_i) \leqslant 1$, $n(x) \leqslant ln(xl^{-1}) \leqslant l[n(x_1\, l^{-1}) + n(x_2\, l^{-1}) + \ldots + n(x_{2k}\, l^{-1})] \leqslant l\, 2^k$. Let $\varrho(x) = r + l$, where $r \geqslant 1$, $0 < l \leqslant 1$. There exist x_1, x_2, $x = x_1 + x_2$ such that $\varrho(x_1) \leqslant r\varrho(x)^{-1}\varrho(x) = r$, $\varrho(x_2) \leqslant (1 - r\varrho(x)^{-1})\varrho(x) = l\varrho(x)^{-1}\varrho(x) = l$. It is shown by induction that a decomposition $x = x_1 + x_2 + \ldots + x_r + x_0$ exists such that $\varrho(x_i) \leqslant 1$, $\varrho(x_0) \leqslant l \leqslant 1$. Hence it follows

$$n(x) \leqslant n(x_1) + n(x_2) + \ldots + n(x_r) + n(x_0) \leqslant rk + k \leqslant k(\varrho(x) + 1).$$

1.9. If ϱ is a modular in $X(\varrho)$ then a quasinorm $n(x)$ (a distributive functional ξ) in $X(\varrho)$ is called *ϱ-continuous*, if $\varrho(x_n) \to 0$ implies $n(x_n) \to 0$ ($\xi(x_n) \to 0$). A distributive and ϱ-continuous functional is called *ϱ-linear*. Evidently, a ϱ-continuous $n(x)$ is continuous with respect to the norm generated by ϱ (by **1.3**, b)), but not conversely. If the condition B.2 is satisfied, then $\varrho(x_n) \to 0$ implies $\|x_n\|_\varrho \to 0$ and then the ϱ-continuity is equivalent to the continuity with respect to $\| \ \|_\varrho$. A set $X^0 \subset X(\varrho)$ is called *modular-dense* in $X(\varrho)$, if to every $x \in X(\varrho)$ there exist $x_n \in X(\varrho)$ and $\lambda > 0$ such that $\varrho(\lambda(x_n - x)) \to 0$.

1.91. If ϱ is a modular satisfying the condition (D) and if there exists an s-homogeneous norm in $X(\varrho)$ continuous with respect to $\| \ \|_\varrho$, then there is a constant $c > 0$ for which

$$\liminf_{t \to \infty} \varrho(tx)\, t^{-s} \geqslant cn(x) > 0 \quad \text{for} \quad x \in X(\varrho).$$

If n is the norm satisfying the assumptions, then $n(x) \leqslant k(t^{-s} + \varrho(tx)\, t^{-s})$, by **1.81** and **1.6**, (b).

1.92. If ϱ is a modular satisfying the condition (D) and if

$$\liminf_{t \to \infty} \varrho(tx)\, t^{-1} = 0$$

in a set modular-dense in $X(\varrho)$, then there exists only a trivial ϱ-linear functional in $X(\varrho)$ ([3]).

We put $n(x) = |\xi(x)|$; then, by **1.91** with $s = 1$, we get $\xi(x) = 0$ in a set modular-dense in $X(\varrho)$. Hence the ϱ-continuity implies $\xi(x) = 0$ always in $X(\varrho)$.

2. In this section the term *φ-function* will mean a function continuous and nondecreasing for $u \geqslant 0$, vanishing only at $u = 0$ and tending to ∞ as $u \to \infty$. If $\varphi(\alpha u + \beta v) \leqslant \alpha^s \varphi(u) + \beta^s \varphi(v)$, $0 < s \leqslant 1$, $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$, then this φ-function will be called *s-convex* (it is an s-convex modular in the space of real numbers). E.g. $\varphi(u) = \psi(u^s)$, where ψ is a convex φ-function, is s-convex.

2.1. We shall consider the following examples of modular spaces:

A. Let μ denote a finite, σ-additive measure, defined on a σ-algebra F of subsets of a set E, X — the space of functions μ-measurable in E, $\varrho(x) = \int_E \varphi(|x(t)|)\, d\mu$, $L^\varphi(\mu) = X^*(\varrho) = X(\varrho)$.

B. X — the space of sequences $x = \{t_\nu\}$, $\varrho(x) = \sum \varphi(|t_\nu|)$, $l^\varphi = X^*(\varrho) = X(\varrho)$.

C. X — the space the elements of which are classes of bounded functions in $\langle a, b\rangle$; two functions belong to the same class if they differ by a constant. $\varrho(x) = \sup \sum \varphi(|x(t_\nu) - x(t_{\nu-1})|)$, where the supremum is taken over all partitions $\pi : a = t_0 < t_1 < \ldots < t_n = b$, $V_\varphi^* = X^*(\varrho)$ ([1]).

2.2 If φ is an s-convex φ-function, then the modulars defined in A—C are s-convex. By this assumption, the modular in V_φ^* satisfies B. 1, whence a norm generated by ϱ can be introduced in V_φ^*. If the measure μ is nonatomic then the modular defined in A satisfies the condition (D). In general, modulars defined in B by means of arbitrary φ do not satisfy the condition (D), e.g. if $\varphi(t)/t \to 0$ as $t \to \infty$ or if $\varphi(t' + t'') < \varphi(t') + \varphi(t'')$ for $t', t'' > 0$.

2.3. A function $x \in V_\varphi^*$ is called *absolutely continuous with respect to* φ ([1]), if to any $\varepsilon > 0$ there is a $\delta > 0$ such that $\sum \varphi(|x(\beta_i) - x(\alpha_i)|) < \varepsilon$ for all systems of non-overlapping intervals $(\alpha_i, \beta_i) \subset \langle a, b\rangle$ satisfying the inequality $\sum \varphi(\beta_i - \alpha_i) < \delta$. We denote by AC_φ^* the set of all x for which λx is absolutely continuous with respect to φ for a $\lambda > 0$ (two functions which differ only by a constant are considered to be the same element of the space). AC_φ^* is a linear subspace of V_φ^*.

2.4. Let $\varphi(u)/u \to 0$ as $u \to 0$. By this assumption:

(a) the modular ϱ defined in C satisfies the condition B. 1 for $x \in AC_\varphi^*$.

(b) ϱ satisfies the condition (D).

Proof of (a). Denote $V_\varphi(x; \alpha, \beta) = \sup \sum \varphi(|x(t_\nu) - x(t_{\nu-1})|)$, where the supremum is taken over all partitions of the interval $\langle \alpha, \beta\rangle \subset \langle a, b\rangle$, $V_\varphi(x; \alpha, \alpha) = 0$. The definition of ϱ is $\varrho(x) = V_\varphi(x; a, b)$. The partition being given, denote $\mathcal{J}'(x) = \sum' \varphi(|x(t_\nu) - x(t_{\nu-1})|)$, $\mathcal{J}''(x) = \sum'' \varphi(|x(t_\nu) - x(t_{\nu-1})|)$, where the summation is extended over all intervals of the length $< \delta$ in the first sum. and over all intervals of the length $\geqslant \delta$ — in the second one. Let $\varrho(x) < \infty$, $x \in AC_\varphi^*$. Since $\varphi(u)/u < \varepsilon(b - a)^{-1}$ for $\delta > 0$ sufficiently small, we have $\mathcal{J}'(x) \leqslant \varepsilon(b - a)^{-1} \sum'(t_\nu - t_{\nu-1}) \leqslant \varepsilon$. Moreover, there holds the inequality $\mathcal{J}''(x) \leqslant (b - a)\delta^{-1} \varphi(2 \sup_{\langle a, b\rangle} |x(t)|)$, for the number of intervals in this sum is $\leqslant (b - a)\delta^{-1}$. Hence we obtain for $0 \leqslant \alpha \leqslant 1$,

$$\Sigma \varphi(\alpha |x(t_\nu) - x(t_{\nu-1})|) \leqslant \mathcal{I}'(x) + (b-a)\,\delta^{-1}\varphi(2\alpha \sup |x(t)|),$$

$$\varrho(\alpha x) \leqslant \varepsilon + \varepsilon \text{ for } \alpha \text{ sufficiently small.}$$

Proof of (b). Let $0 < \varrho(x) < \infty$, $x \in AC_\varphi^*$. It is easily shown that $V_\varphi(x; a, \tau)$ and $V_\varphi(x; \tau, b)$ are continuous functions of τ for $a \leqslant \tau \leqslant b$. We define the functions

$$x'_\tau(t) = \begin{cases} x(t) \text{ for } a \leqslant t < \tau \\ x(\tau) \text{ for } b \geqslant t \geqslant \tau, \end{cases} \qquad x''_\tau(t) = \begin{cases} x(\tau) \text{ for } a \leqslant t < \tau \\ x(t) \text{ for } b \geqslant t \geqslant \tau. \end{cases}$$

Since $x(t) + x(\tau) = x'_\tau(t) + x''_\tau(t)$ for $t \in \langle a, b\rangle$, we have $x = x'_\tau + x''_\tau$ and moreover, $\varrho(x'_\tau) = V_\varphi(x; a, \tau)$, $\varrho(x''_\tau) = V_\varphi(x; \tau, b)$. Let $0 < \alpha < 1$; there is a τ^*, $a < \tau^* < b$, such that $\varrho(x'_{\tau^*}) = \alpha\varrho(x)$. Since $\varrho(x'_{\tau^*}) + \varrho(x''_{\tau^*}) = V_\varphi(x; a, \tau^*) + V_\varphi(x; \tau^*, b) \leqslant \leqslant V_\varphi(x, a, b) = \varrho(x)$, we have $\varrho(x''_{\tau^*}) \leqslant (1-\alpha)\varrho(x)$.

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REFERENCES

[1] J. Musielak and W. Orlicz, *On generalized variations* (I), Studia Math., **18** (1959), 11—41.

[2] — — *On modular spaces*, Studia Math., **18** (1959), 49—65.

[3] — — *Some remarks on modular spaces*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. **7** (1959), 669—676.

[4] H. Nakano, *Topology and linear topological spaces*, Tokyo, 1951.

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MATHEMATICS

Definability of Sets in Models of Axiomatic Theories

by

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In the present paper we introduce and discuss the notion of definability of sets of integers in models of first order theories. This notion is a natural generalization of the notion of representability in a model of type theory [1].*) Our main result states that under very general assumptions the family of sets which are definable in all models of a theory coincides with the family of recursive sets. **)

1. The space of models. Let T be a consistent first order theory with a finite or denumerably infinite number of predicates and with a denumerable number of variables $x_1, x_2, \ldots$ The propositional connectives are noted as $\sim$, $\succ$, $\vee$, $\wedge$, and the quantifiers as $\bigvee$ and $\bigwedge$.

A *frame* $\mathfrak{M}$ of T (in the set of all integers) is a family of relations indexed by the predicates of T, each relation having the same number of arguments (running over integers) as its index. A frame $\mathfrak{M}$ is a *model* if all theorems of T are true in $\mathfrak{M}$. We denote by M the family of all models of T. From now on until section 4 the word "model" will be synonymous with the expression "an element of M".

If φ is a formula, then $\vdash_T \varphi$ means that φ is provable in T. Obviously we have then $\vdash_T \bigwedge x_i x_j \ldots x_k \varphi$ for arbitrary $i, j, \ldots, k$ as well as $\vdash_T \mathrm{Sb}(x_i/x_j)\varphi$, where $\mathrm{Sb}(x_i/x_j)\varphi$ is a formula obtained from φ by replacing the free occurrences of x_i by x_j and the bound occurrences of x_j (if any) by x_{j+k} where k is the largest index of a variable occurring in φ.

For $\mathfrak{M}$ in M we write $\models_{\mathfrak{M}} \varphi$ if φ is satisfied in $\mathfrak{M}$ by an assignment which correlates the value j to the variable x_j, $j = 1, 2, \ldots$

We put $[\varphi] = \{\mathfrak{M}: \models_{\mathfrak{M}} \varphi\}$. It is known that set theoretical operations on classes $[\varphi]$ correspond to logical operations on formulas, e.g., $[\varphi] \cup [\psi] = [\varphi \vee \psi]$.

We shall consider M as a topological space by taking sets $[\varphi]$ as a basis for open sets; φ runs here over all formulas of T.

*) Parts of our discussion (especially those concerned with the ω-standard models) were stimulated by the reading of this paper.

**) A part of our results was announced in [4].

THEOREM 1. *M is 0-dimensional separable space which admits a complete metric (i.e., M is an absolute G_δ).*

Proof. The only non-trivial part of the theorem is that which states the existence of a complete metric. We shall sketch a short but indirect proof based on ideas of [6].

Formulas of T form a Boolean algebra under the usual operations, formulas mutually equivalent in T being treated as a single element of the algebra. The family of all prime filters of this algebra (i.e., its Stone space) is a bicompact 0-dimensional separable topological space. The open basis of this space consists of sets $\{\varphi\}$ of prime filters containing φ. Our space M is homeomorphic with the set of those prime filters which preserve denumerable meets corresponding to general quantifiers. More exactly these filters $\mathfrak{f}$ have the property: whenever φ is a formula and $\mathrm{Sb}(x_i/x_n)\,\varphi$ is in $\mathfrak{f}$ for $n = 1, 2, \ldots$, then so is $\bigwedge x_i\,\varphi$. Since the set of all formulas is denumerable, filters $\mathfrak{f}$ with the above property form indeed a G_δ set. (In [6] it is proved, moreover, that this set is residual which immediately yields the completeness theorem). In order to conclude our proof it is sufficient to use the well-known fact that every G_δ set contained in a complete metric space possesses a complete metric (see e.g., [3], p. 316).

2. ω-models. Let π_n, $n = 0, 1, 2, \ldots$ be a sequence of formulas with exactly one free variable x_1. We denote the formula $\mathrm{Sb}(x_1/x_j)\,\pi_n$ by $\pi_n(x_j)$ and assume that $\vdash_T \pi_n \to \pi_0$ for $n = 1, 2, \ldots$

We call T an *ω-closed* theory (with respect to the sequence π_n) if for every formula φ the condition that $\vdash_T \pi_n \to \varphi$ for $n = 1, 2, \ldots$ implies that $\vdash_T \pi_0 \to \varphi$.

We call $\mathfrak{M}$ an *ω-model* (with respect to the sequence π_n) if for every formula φ with at most free variable x_1 the condition $\models_{\mathfrak{M}} \bigvee x_1 (\pi_0 \wedge \varphi)$ implies the existence of an integer $n \geqslant 1$ such that $\models_{\mathfrak{M}} \bigvee x_1 (\pi_n \wedge \varphi)$.

We call $\mathfrak{M}$ an *ω-standard model* (with respect to the sequence π_n) if for every integer $i \geqslant 1$ there is an integer $n \geqslant 1$ such that $\models_{\mathfrak{M}} \pi_0(x_i) \leftrightarrow \pi_n(x_i)$.

The following obvious theorem and the (slightly less obvious) Corollary 3 given below clarify the connection between ω-models and ω-standard models. Write $\mathfrak{M}_1 \sim \mathfrak{M}_2$ if for every φ without free variables the conditions $\models_{\mathfrak{M}_1} \varphi$ and $\models_{\mathfrak{M}_2} \varphi$ are equivalent (i.e. $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are elementarily equivalent, [7]).

THEOREM 2. *If $\mathfrak{M}_1 \sim \mathfrak{M}_2$ and $\mathfrak{M}_1$ is an ω-standard model, then $\mathfrak{M}_2$ is an ω-model: in particular every ω-standard model is an ω-model.*

THEOREM 3. *If T is ω-closed, then the set M_ω of its ω-standard models is a residual G_δ-set in M.*

Proof. Since $M_\omega = \bigcap_{i=1}^{\infty} \bigcup_{n=1}^{\infty} [\pi_0(x_i) \leftrightarrow \pi_n(x_i)]$ it is sufficient to show that each of the open sets $\bigcup_{n=1}^{\infty} [\pi_0(x_{i_0}) \leftrightarrow \pi_n(x_{i_0})]$, $i_0 = 1, 2, \ldots$ is dense in M, i.e., that for no φ this set is disjoint with $[\varphi]$ unless $[\varphi] = 0$. Otherwise we would have $0 = [\varphi] \cap [\pi_0(x_{i_0}) \to \pi_n(x_{i_0})] = [\varphi \wedge \sim \pi_0(x_{i_0})] \cup [\varphi \wedge \pi_n(x_{i_0})]$ for $n = 1, 2, \ldots$ whence we would obtain for every $\mathfrak{M}$ in M

$$\models_{\mathfrak{M}} \varphi \to \pi_0(x_{i_0}), \quad \models_{\mathfrak{M}} \pi_n(x_{i_0}) \to \sim \varphi \text{ for } n = 1, 2, \ldots$$

Using completeness theorem and the assumption that T is ω-closed we would further obtain $\vdash_T \varphi \to \pi_0(x_{i_0})$ and $\vdash_T \pi_0(x_{i_0}) \to \sim \varphi$, whence $\vdash_T \sim \varphi$ and $[\varphi]$ would be void.

COROLLARY 1. *If T is ω-closed, then $M_\omega \neq 0$.* *)

Indeed, the space M being complete and hence of second category on itself. the void set is not residual.

COROLLARY 2. *An ω-closed theory has a complete consistent and ω-closed extension.*

Proof. Take the set of formulas without free variables which are true in an ω-standard model of T.

COROLLARY 3. *For every ω-model $\mathfrak{M}$ of T there is an ω-standard model $\mathfrak{M}'$ such that $\mathfrak{M} \sim \mathfrak{M}'$.*

Proof. The set of formulas φ without free variables which are true in $\mathfrak{M}$ is a consistent ω-closed extension of T. According to Corollary 1 this extension has an ω-standard model $\mathfrak{M}'$. Obviously $\mathfrak{M} \sim \mathfrak{M}'$.

3. Representability and definability. A set Z of integers is representable in T (with respect to a sequence π_n) if there is a formula φ with exactly one free variable x_1 such that

$n \in Z$ *implies* $\vdash_T \bigvee x_1 (\pi_n \wedge \varphi)$,

n *non* $\in Z$ implies $\vdash_T \sim \bigvee x_1 (\pi_n \wedge \varphi)$.

Z is *definable* in a model $\mathfrak{M}$ by a formula φ (which may contain arbitrary free variables) if $n \varepsilon Z$ is equivalent to $\models_{\mathfrak{M}} \bigvee x_1 (\pi_n \wedge \varphi)$ for $n = 1, 2, \ldots$

A theory which is consistent and which results from T by adjunction of a finite number of individual constants and of a finite number of axioms is called a *finite extension* of T.

THEOREM 3. *A representable set is definable in every model of T.*

Less obvious is the following

PRINCIPAL LEMMA. *Given an arbitrary set Z of integers. the set M_Z of models in which Z is definable is an F_σ set in M; if M_Z is not of first category in M, then there is a finite extension of T in which Z is representable.*

Proof. M_Z is an F_σ since

$$M_Z = \bigcup_\varphi \Big(\bigcap_{n \varepsilon Z} \{\mathfrak{M}: \models_{\mathfrak{M}} \bigvee x_1 (\pi_n \wedge \varphi)\} \cap \bigcap_{n \varepsilon Z} \{\mathfrak{M}: \models_{\mathfrak{M}} \sim \bigvee x_1 (\pi_n \wedge \varphi)\} \Big) =$$

$$= \bigcup_\varphi \bigcap_{n=1}^{\infty} \{\mathfrak{M}: n \varepsilon Z \equiv \models_{\mathfrak{M}} \bigvee x_1 (\pi_n \wedge \varphi)\} = \bigcup_\varphi X_\varphi .$$

If M_Z is not of first category, then one of the sets X_φ must have a non-void interior. Let $[\psi]$ be a non void element of the open basis of M which is contained in X_φ. For every n and every $\mathfrak{M}$ satisfying $\models_{\mathfrak{M}} \psi$ we have either $n \in Z$ and

*) This corollary was first proved in [5] in essentially the same way. We believe that the use of the topological language makes the proof more conspicuous.

$\mathfrak{M} \vdash \bigvee x_1 (\pi_n \wedge \varphi)$ or $n \notin Z$ and $\vdash_{\mathfrak{M}} \sim \bigvee x_1 (\pi_n \wedge \varphi)$. Applying the completeness theorem we obtain

$$\vdash_T \psi \to \bigvee x_1 (\pi_n \wedge \varphi) \text{ for } n \in Z,$$

$$\vdash_T \psi \to \sim \bigvee x_1 (\pi_n \wedge \varphi) \text{ for } n \notin Z.$$

Now add to T as many constants as there are free variables in the formula $\psi \to \bigvee x_1 (\pi_n \wedge \varphi)$ and an axiom resulting from ψ by replacing its free variables by the constants. In the resulting finite extension of T the set Z is representable. Note that the extension is consistent since $[\psi]$ is not void. This concludes the proof of the principal lemma.

4. Final results. A model $\mathfrak{M}$ of T (which may or may not be an element of M) is called ω-standard with respect to a sequence π_n of formulas with exactly one free variable x_1 if the interpretation of π_0 in $\mathfrak{M}$ is the set theoretical union of the interpretations of π_n in $\mathfrak{M}$.

A set Z of integers is *definable* in $\mathfrak{M}$ (with respect to the sequence π_n) if there is a formula φ and an assignment f of values in $\mathfrak{M}$ to the free variables of φ such that f satisfies in $\mathfrak{M}$ the formula $\bigvee x_1 (\pi_n \wedge \varphi)$ if and only if $n \in Z$.

It is easy to see that if $\mathfrak{M}$ is in M, then these definitions coincide with those given in sections 2 and 3.

We denote by $\mathfrak{F}$ (by $\mathfrak{F}_\omega$) the family of sets Z which are definable in all models (all ω-standard models) of T. By $\mathfrak{F}'$ (resp. $\mathfrak{F}'_\omega$) we denote the family of sets Z which are definable in all models $\mathfrak{M}$ in M (resp. in M_ω). Finally we denote by $\mathfrak{F}_0$ the family of sets representable in T.

LEMMA. *$\mathfrak{F} = \mathfrak{F}'$ and $\mathfrak{F}_\omega = \mathfrak{F}'_\omega$.*

Proof. The inclusions $\mathfrak{F} \subseteq \mathfrak{F}'$ and $\mathfrak{F}_\omega \subseteq \mathfrak{F}'_\omega$ being obvious it is sufficient to show that if Z is definable in all denumerable models (ω-models) it is definable in an arbitrary model (ω-model) $\mathfrak{M}$. Let $\mathfrak{M}'$ be a denumerable sub-model of $\mathfrak{M}$ such that $\mathfrak{M}$ is an arithmetical extension of $\mathfrak{M}'$ (cf. [8]). The lemma follows from the remark that if $\mathfrak{M}$ is an ω-standard model, then so is $\mathfrak{M}'$, and that if Z is definable in $\mathfrak{M}'$, then it is definable in $\mathfrak{M}$.

COROLLARY 4. *If T is axiomatizable and all recursive sets are representable in T, then the families $\mathfrak{F}_0$ and $\mathfrak{F}$ coincide with the family $\mathfrak{R}$ of recursive sets.*

Proof. It is obvious that $\mathfrak{R} \subseteq \mathfrak{F}_0$. By the principal lemma and Theorem 1 the condition $Z \in \mathfrak{F}'$ implies that Z is representable in a finite extension of T and hence (by the axiomatizability of T) recursive.

COROLLARY 5. *If T is ω-closed, all hyper-arithmetic sets are representable in T and if the set of theorems of T is a Π^1_1 set, then $\mathfrak{F}_0$ and $\mathfrak{F}_\omega$ coincide with the family HA of hyper-arithmetic sets.*

Proof. The equation $\mathfrak{F}_0 = HA$ is well-known (cf. [2]) and the inclusion $\mathfrak{F}'_\omega \subseteq \mathfrak{F}_0$ is proved as above using Theorem 3 instead of Theorem 1.

Call a theory T *persistent with respect to representability* if sets representable in a finite extension of T are representable already in T. From the Principal Lemma we directly obtain

COROLLARY 6. *If T is persistent with respect to representability, then $\mathfrak{F}_0 = \mathfrak{F}$. If moreover T is ω-closed, then $\mathfrak{F}_0 = \mathfrak{F}_\omega$.*

These corollaries can be applied, e.g., to theories A and A_ω of [2], if we modify them so as to render them theories of first order. To achieve this we drop the distinction between number and function variables and introduce two new predicates $P_0(x)$ ("x is an integer") and $F_0(x)$ ("x is a function") as well as one new operation symbol denoting the application of a function to an argument. Axioms of A are modified by appropriate relativizations of quantifiers. As π_0 we take the formula $P_0(x_1)$ and as π_n $(n \geqslant 1)$ the formula $x_1 = n$.

In conclusion we mention an application to an axiomatic system T_0 of set theory. To fix the ideas let us assume that T_0 is the Zermelo Fraenkel axiomatic system. Let π_0 be a formula of T_0 expressing the property of being an integer and π_n $(n \geqslant 1)$ the formula expressing the property of being the n-th integer. Modifying slightly the definition given in [1] we shall say that a set Z of integers is representable in a model $\mathfrak{M}$ of T_0 if there is an element m in $\mathfrak{M}$ such that n is in Z if and only if the formula $\bigvee x_1 (\pi_n \wedge (x_1 \varepsilon x_2))$ is satisfied in $\mathfrak{M}$ by m. From the Corollaries 4 and 5 we obtain.

COROLLARY 7. *Sets which are representable in all models of T_0 are recursive; sets which are representable in all ω-standard models of T_0 are hyper-arithmetic.*

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REFERENCES

[1] O. Gandy, G. Kreisel, W. Tait, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **8** (1960), 577—582.

[2] A. Grzegorczyk, A. Mostowski, Cz. Ryll-Nardzewski, Jour. Symbol. Logic, **23** (1958), 188—206.

[3] K. Kuratowski, *Topologie I*, Monogr. Matem., Warszawa—Wrocław, 1948.

[4] A. Mostowski, Cz. Ryll-Nardzewski, Jour. Symbol Logic, **23** (1958), 458—459.

[5] S. Orey, Jour. Symbol. Logic, **21** (1956), 246—252.

[6] H. Rasiowa, R. Sikorski, Fund. Mathem., **37** (1950), 193—200.

[7] A. Tarski, Proc. Int. Congress of Math., **1** (1950), 705—720.

[8] A. Tarski, R. L. Vaught, Compositio Math., **13** (1957), 81—102.

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A Linear Iterative Method with a Vector Parameter

by

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1. This paper concerns an iterative method of solving linear equations in Hilbert space. In particular, we obtain an iterative method for the solution of a system of linear algebraic equations. The advantage of the method presented here is the changeability of its convergence speed. This fact takes place because the method contains an arbitrary vector as a parameter and the speed depends actually on this vector. Thus, changing this vector, we change the speed of the convergence at the same time. It happens frequently that the convergence is very slow, especially when the operator or the matrix are ill conditioned. In this case we can try to improve the convergence replacing this vector by an arbitrary one. Notice that the parameter vector of the method does not mean the arbitrary initial approximate solution which exerts propably influence on the convergence as well as in many methods, however this fact is not visible from the method itself. From this point of view the method presented here gives a kind of over-relaxation, where the numerical parameter is replaced by an arbitrarty vector. The convergence of our method seems to be faster than that of the method of steepest descent investigated by L. V. Kantorovitch [1].

Let A be a linear (i.e. additive and homogeneous) operator with domain and range in a Hilbert space H. Consider the linear equation

$$Ax = b, \qquad x, b \in H. \tag{1}$$

We shall assume operator A to be positive definite and selfadjoint, or, more precisely,

$$m(x, x) \leqslant (Ax, x) \leqslant M(x, x),$$

where $0 < m < M < +\infty$, and

$$A^* = A,$$

where A^* is the adjoint of A.

The general equation can be reduced to this case.

Without loss of generality one can suppose that $\|b\| = 1$.

Instead of Eq. (1) we shall consider the following equation

$$Ax = (Ax, b)\, b, \quad x \neq 0, \quad \|b\| = 1. \tag{2}$$

Thus, if x is a solution of Eq. (2), then $x/(Ax, b)$ is the solution of Eq. (1). It follows from (2) that $(Ax, b) \neq 0$, since operator A is non-singular.

We shall now define the iterative process of solving Eq. (2).

Denote by R the linear operator defined as follows:

$$Rx = Ax - (Ax, b)\, b, \quad x \in H, \quad \|b\| = 1. \tag{3}$$

Let x_0 be an arbitrary approximate solution of Eq. (2) such that

$$(x_0, b) \neq 0. \tag{4}$$

Then

$$x_{n+1} = x_n - \alpha_n Rx_n, \tag{5}$$

where

$$z_n = Rx_n, \quad \alpha_n = \frac{(z_n, Az_n)}{\|Rz_n\|^2}, \tag{6}$$

and, by (3),

$$\|Rx\|^2 = \|Ax\|^2 - (Ax, b)^2. \tag{7}$$

It is easy to see that, by (3) and (6),

$$z_{n+1} = z_n - \alpha_n Rz_n.$$

Thus, we have

$$\|z_{n+1}\|^2 = \|z_n\|^2 - 2\alpha_n [(Az_n, z_n) - (Az_n, b)(b, z_n)] + \alpha_n^2 [\|Az_n\|^2 - (Az_n, b)^2].$$

Making use of the fact that

$$(z_n, b) = 0 \text{ for } n = 0, 1, 2, \ldots,$$

we get

$$\|z_{n+1}\|^2 = \|z_n\|^2 - \frac{(z_n, Az_n)^2}{\|Rz_n\|^2}.$$

We shall now show that the sequence of z_n converges to zero. We have

$$\frac{\|z_{n+1}\|^2}{\|z_n\|^2} = 1 - \frac{(z_n, Az_n)^2}{\|z_n\|^2 \|Rz_n\|^2} = 1 - \frac{(z_n, Az_n)^2}{\|z_n\|^2 \|Az_n\|^2} \cdot \frac{\|Az_n\|^2}{\|Rz_n\|^2}.$$

Hence, we get

$$\|z_{n+1}\|^2 = \left[1 - \frac{(z_n, Az_n)^2}{\|z_n\|^2 \|Az_n\|^2} \left(1 + \frac{(Az_n, b)^2}{\|Az_n\|^2 - (Az_n, b)^2}\right)\right] \|z_n\|^2. \tag{8}$$

One can show that

$$\inf_{\|z\|=1} \frac{(z, Az)^2}{\|Az\|^2} = \frac{4mM}{(M+m)^2}. \tag{9}$$

This has been proved by Krasnoselsky [2] in the case of the matrix A. It follows from (8) and (9) that

$$\|z_{n+1}\|^2 \leqslant \left[1 - q\left(1 + \frac{(Az_n, b)^2}{\|Az_n\|^2 - (Az_n, b)^2}\right)\right] \|z_n\|^2, \tag{10}$$

where

$$q = \frac{4mM}{(M+m)^2}.$$

Relation (10) shows that

(11) $$z_n \to 0 \text{ as } n \to \infty.$$

It is easy to see that

(12) $$\frac{\inf}{n} |(Ax_n, b)| = c > 0, c \approx m(x_0, b)$$

In fact, we have, by (12)

$$\|z_n\|^2 = \|Ax_n\|^2 - (Ax_n, b)^2 \to 0 \text{ as } n \to 0.$$

On the other hand, we infer that

(13) $$\frac{\inf}{n} \|x_n\| > 0,$$

since, by (5) and (6), we have

$$x_n = x_0 - \alpha_0 z_0 - \alpha_1 z_1 - \ldots - \alpha_{n-1} z_{n-1}.$$

If (13) were not true, we should get

$$(x_0, b) = (x_{n_k}, b) \to 0 \text{ as } k \to \infty,$$

since $(z_i, b) = 0$ for $i = 0, 1, 2, \ldots$ Thus, we obtain a contradiction to (4). Consequently, relation (12) is true in virtue of (13) and the non-singularity of A.

Now, if for x_n defined in (5) the norm of $z_n = Rx_n$ is sufficiently small, in virtue of (11), then we take

(14) $$y_n = \frac{x_n}{(Ax_n, b)} = \frac{x_n}{(x_n, Ab)}$$

as the approximate solution of Eq. (1). Then the error estimate is given by the following formula

(15) $$\|y_n - x^*\| \leqslant \frac{\|z_n\|}{m(x_n, Ab)} \leqslant \frac{\|z_n\|}{mc},$$

where x^* is the solution of Eq. (1). In fact, we have

$$\|z_n\| = \|Ax_n - (Ax_n, b)\, b\| - |(Ax_n, b)|\, \|Ay_n - b\| \geqslant mc\, \|y_n - x^*\|.$$

Relation (10) shows that the convergence is faster than a geometric progression with ratio

(16) $$\bar{q} = (1 - q)^{1/2} = \frac{M - m}{M + m}.$$

Kantorovitch [1] has shown that the method of steepest descent is at least as fast as a geometric progression with ratio q defined by (16). Thus, we conclude that the method defined by (5) and (14) gives in general a faster convergence than the method of steepest descent.

We shall now consider a particular case of our method. Suppose that operator A is given by a matrix.

Let us assume that

$$(Az_n, b) = 0 \text{ for } n \geqslant k.$$

In this case we get

$$z_{n+1} = z_n - \alpha_n Az_n, \text{ for } n \geqslant k.$$

Since

$$Az_{n+1} = Az_n - \alpha_n A^2 z_n,$$

we obtain

$$(A^2 z_n, b) = 0 \text{ for } n \geqslant k.$$

Hence, we get by induction

$$(A^i z_k, b) = 0 \text{ for } i = 0, 1, 2, \ldots,$$

or, equivalently,

$$(z_k, A^i b) = 0 \text{ for } i = 0, 1, 2, \ldots$$

If the vectors $A^i b$ form a basis in the space, then

$$z_k = 0.$$

Thus, the number of iterations in this particular case is finite.

2. We shall now introduce an arbitrary vector as a parameter in the method defined in (5) and (14). Relation (10) shows that the convergence of the sequence of z_n depends on the vector b. Thus, we can make use of this fact.

Let y be an arbitrary fixed vector of H. If x^* is the solution of Eq. (1) then, obviously, we have

$$A(x^* + y) = Ay + b.$$

Putting

$$b' = \frac{Ay + b}{\|Ay + b\|}, \quad x = \frac{x^* + y}{\|Ay + b\|}$$

we obtain the equation

$$Ax = b'$$

instead of Eq. (1).

Then Eq. (2) has the form

$$Ax = (Ax, b')\, b', \quad x \neq 0, \quad \|b'\| = 1 \tag{17}$$

or

$$\|Ay + b\|^2 Ax = (Ax, Ay + b)(Ay + b).$$

Thus, the solution of Eq. (1) is

$$x^* = \frac{\|Ay + b\|^2 x}{(Ax, Ay + b)} - y,$$

where x is a non-trivial solution of Eq. (17).

To obtain the approximate solution of Eq. (17) we shall apply the same procedure as in (5). Let us start with an initial approximate solution x_0 of Eq. (17) such that

$$(x_0, Ay + b) \neq 0.$$

This condition corresponds to that in (4). The sequence of approximate solutions x_n of Eq. (17) is defined by the following recurrent formula

$$x_{n+1} = x_n - \alpha_n R_1(x_n),$$

where

$$R_1(x) = Ax - \frac{(Ax, Ay + b)}{\|Ay + b\|^2}(Ay + b)$$

and

$$\alpha_n = \frac{(z_n, Az_n)}{\|R_1(z_n)\|^2}, \qquad z_n = R_1(x_n).$$

Thus, we obtain

$$x_{n+1} = x_n - \frac{(z_n, Az_n)\,\|Ay+b\|^2}{\|Ay+b\|^2\,\|Az_n\|^2 - (Az_n, Ay+b)^2}\, z_n,$$

where

$$z_n = Ax_n - \frac{(Ax_n, Ay+b)}{\|Ay+b\|^2}(Ay+b). \tag{18}$$

Further, we have

$$z_{n+1} = z_n - \frac{(z_n, Az_n)\,\|Ay+b\|^2}{\|Ay+b\|^2\,\|Az_n\|^2 - (Az_n, Ay+b)^2}\left[Az_n - \frac{(Az_n, Ay+b)}{\|Ay+b\|^2}(Ay+b)\right].$$

Since the sequence of z_n converges to the zero element of H, the norm of z_n will be sufficiently small for large n. Then we take

$$y_n = \frac{x_n\,\|Ay+b\|^2}{(Ax_n, Ay+b)} - y = \frac{x_n\,\|Ay+b\|^2}{(x_n, A(Ay+b))} - y \tag{19}$$

as the corresponding approximate solution of Eq. (1).

The rate of convergence is now given by the following formula

$$\|z_{n+1}\|^2 \leqslant \left[1 - q\left(1 + \frac{(Az_n, Ay+b)^2}{\|Ay+b\|^2\,\|Az_n\|^2 - (Az_n, Ay+b)^2}\right)\right]\|z_n\|^2$$

which replaces relation (10) and q is here the same as in (10).

We shall now calculate the error estimate for the approximate solution of Eq. (1). We have

$$\|z_n\| = \frac{|(Ax_n, Ay+b)|}{\|Ay+b\|^2}\left\|A\left(\frac{x_n\,\|Ay+b\|^2}{(Ax_n, Ay+b)} - y - x^*\right)\right\| \geqslant \frac{|(Ax_n, Ay+b)|}{\|Ay+b\|^2} m\,\|y_n - x^*\|$$

in virtue of (18) and (19). Hence, we obtain

$$\|y_n - x^*\| \leqslant \frac{\|Ay+b\|^2}{|(Ax_n, Ay+b)|\,m}\|z_n\| \leqslant \frac{\|Ay+b\|^2}{cm}\|z_n\|,$$

where

$$c = \inf_n |(Ax_n, Ay+b)| > 0$$

and x^* is the solution of Eq. (1).

All assertions of section 1 remain also true in this case.

The main results reported in the present paper have been obtained by the author in January, 1960, at the New York University.

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REFERENCES

[1] L. V. Kantorovitch, *Functional analysis and applied mathematics* [in Russian]. Uspekhi Matem. Nauk (NS) **3** (1948) No. 6, 89—185 [translated into English: National Bureau of Stand.].

[2] M. A. Krasnoselsky and S. G. Krein, *Iteration process with minimum residua* [in Russian]. Matem. Sbornik (NS), **31** (1952), 157—161

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MATHEMATICS

An Iterative Method for Linear Algebraic Equations

by

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Presented by W. ORLICZ on February 7, 1961

This paper contains an iterative method for the approximate solution of a system of linear algebraic equations. Actually, we consider the solution of this system as an eigenproblem. Then we can solve this eigenproblem by the well-known power method. However, the main point is that we introduce an arbitrary vector as a parameter. Thus, the speed of the convergence of the method depends on an arbitrary vector. Hence, we obtain a certain kind of over-relaxation, where we have an arbitrary vector instead of a numerical parameter. It happens frequently that the matrix of the system of equations is ill conditioned. Then the iterative method may be convergent very slowly. In this case we can try to improve the convergence by changing the vector parameter.

Another iterative method depending on a vector parameter is given in [1].

1. Consider the linear equation

$$Ax = b, \quad b, x \in R_n, \tag{1}$$

where A is a linear operator with domain and range in the Euclidean space R_n, defined by the matrix

$$(a_{ik}), \quad i, k = 1, 2, \ldots, n.$$

Suppose $\|b\| = 1$ and write the system (1) homogeneously as follows

$$Bx = 0, \quad x \neq 0, \tag{2}$$

where

$$Bx = Ax - (Ax, b)\, b.$$

Let us assume that A is non-singular. Then, of course,

$$y = \frac{z}{(Ax, b)} \tag{3}$$

is a solution of Eq. (1) if x is a solution of Eq. (2). Operator B is then defined by the following matrix

$$(a_{ik} - c_k b_i), \quad i, k = 1, 2, \ldots, n,$$

where

$$c_k = (a_k, b), \quad a_k = (a_{1k}, a_{2k}, \ldots, a_{nk})$$

and

$$b = (b_1, b_2, \ldots, b_n).$$

Premultiply Eq. (2) by B', where

$$B'z = A'z - (b, z)\, A'b.$$

Then we obtain

$$Tx = B'Bx = A'Ax - (Ax, b)\, A'b = 0,$$

since b is orthogonal to Bx, i.e.,

$$(b, Bx) = 0,$$

A' being the transposed matrix. This may be interpreted as an eigenequation:

$$Tx = \lambda x,$$

when $\lambda = 0$ is known. $T = B'B$ is definite and all its eigenvalues are $\geqslant 0$. $\lambda_n = 0$ being the smallest.

We shall now apply the power method to the matrix T, i.e., the iteration of T on a vector. But we determine the approximate dominant eigenvalue λ_1 of T by a few iterations only.

Let λ_1' be this approximate dominant eigenvalue. Then we consider the matrix

$$T_1 = T - \lambda_1' I,$$

where I denotes the identity matrix, and the dominant (negative) eigenvalue of T_1 is $-\lambda_1'$. The corresponding eigenvector is x and has to be computed by the power method. Then the solution of Eq. (1) is given by formula (3).

Remark. Using the power method we get a considerable simplification of the calculation. For it is unnecessary to compute $T = B'B$ which would involve n^3 multiplications. One rather iterates T_1 on the arbitrary vector in the following manner:

$$\begin{aligned} T_1 z &= Tz - \lambda_1' z \\ &= A'(Az) - (Az, b)\, A'b - \lambda_1' z \\ &= A'u - (Az, b)\, A'b - \lambda_1' z = v, \end{aligned}$$

where $u = Az$.

With v the iteration is repeated in the same way.

2. We shall now introduce an arbitrary vector y as a vector parameter. Thus, we consider the equation

(4) $$Ax = b'$$

instead of Eq. (1), where

$$b' = \frac{Ay + b}{\|Ay + b\|}.$$

Further, we replace Eq. (2) by

(5) $$B_1 x = 0, \quad x \neq 0,$$

where

$$B_1 x = Ax - \frac{(Ax, Ay+b)}{\|Ay+b\|^2}(Ay+b).$$

If x is a solution of Eq. (5), then

$$z = \frac{x\|Ay+b\|^2}{(Ax, Ay+b)} - y \tag{6}$$

is a solution of Eq. (1). We have then, of course,

$$(Ax, Ay+b) \neq 0 \text{ for } Ay+b \neq 0,$$

since A is non-singular by the hypothesis. In the same way as in sec. **1** we determine

$$T_1 x = B_1' B_1 x = A' Ax - \frac{(Ax, Ay+b)}{\|Ay+b\|^2} A'(Ay+b) = 0.$$

This may be considered as an eigenequation

$$T_1 x = \lambda x,$$

when $\lambda = 0$ is known.

We determine the approximate dominant eigenvalue λ_1 of T_1 by a few iterations only. Suppose that λ_1' is the approximate dominant eigenvalue of T_1 obtained in this way. Then we consider the matrix

$$T_1 - \lambda_1' I$$

and we apply the power method to compute the eigenvector x corresponding to the dominant (negative) eigenvalue $-\lambda_1'$ of this matrix. The vector x being determined, we get the solution z of Eq. (1) by the formula in (6).

The Remark of sec. **1** remains true also in this case.

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REFERENCES

[1] M. Altman. *A linear iterative method with a vector parameter*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **9** (1961) [this issue].

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Electronic Spectra of Dimers. I. Nature of the Coupling

by

A. WITKOWSKI

Presented by J. WEYSSENHOFF on December 8, 1960

1. Consider a dimer composed of two identical moieties A, B. Denote the molecular Hamiltonian of monomer A by H_A, electron co-ordinates by q_A and nuclear co-ordinates by Q_A. Then

$$H_A = T_A + V_A, \tag{1}$$

where T_A represents the kinetic nergy of nuclei and V_A governs the electronic motion in the field of stationary nuclei. The Hamiltonian H for the dimer equals:

$$H = H_A + H_B + V_{AB} + T_L, \tag{2}$$

where V_{AB} is the mutual potential energy of the moieties and T_L the kinetic energy of the relative motions of their centres of gravity (motion described by co-ordinate L). When both moieties of the dimer are in the ground electronic states N, then the ground electronic state of the dimer is described by the function π_0:

$$\pi_0 = \psi_N(q_A, Q_A)\,\psi_N(q_B, Q_B) \quad \text{and} \quad V_A\,\psi_N(q_A, Q_A) = V_N(Q_A)\,\psi_N(q_A, Q_A), \tag{3}$$

where characteristic value $V_N(Q_A)$ provides the potential energy surface for the nuclear motions, denoted by the index N for ground electronic state and by M for an excited state. In this notation the vibrational motions of the dimer are described [1] by the Hamiltonian h

$$h = h_A + h_B + h_L,$$

where

$$\begin{cases} h_A = T_A + V_N(Q_A), & h_B = T_B + V_N(Q_B) \\ h_L = T_L + V_0(L), & V_0(L) = \{\pi_0, V_{AB}\,\pi_0\} \end{cases} \tag{4}$$

and where $\{\,,\}$ denotes integration over electronic co-ordinates.

However, when one of the moieties is electronically excited, the wave function of the dimer takes a more general form

$$\alpha\pi_A + \beta\pi_B, \tag{5}$$

where α, β depend on nuclear co-ordinates Q_A, Q_B, and

$$\begin{cases} \pi_A = \psi_M(q_A, Q_A)\,\psi_N(q_B, Q_B), \\ \pi_B = \psi_N(q_A, Q_A)\,\psi_M(q_B, Q_B). \end{cases} \tag{6}$$

It has been shown in [1] that vibronic states of a dimer composed of two chromophoric moieties are described by the Hamiltonian $\bar{h}$:

$$h = \frac{1}{2}(h'_A + h_A + h'_B + h_B) + h'_L + \frac{1}{2}(h'_A - h_A - h'_B + h_B)\,\sigma_3 + v\sigma_1 . \tag{7}$$

where

$$h'_A = T_A + V_M(Q_A)$$

$$h'_B = T_B + V_M(Q_B)$$

$$h_L^{cc'} = T_L\,\delta_{cc'} + \{\pi_c, V_{AB}\,\pi_{c'}\}; \qquad c, c' = A, B$$

$$h'_L = h_L^{AA} = h_L^{BB}$$

$$h_L^{AB} = h_L^{BA} = v(L)$$

and where σ_1, σ_3 are familiar Pauli matrices, and the eigen-functions Φ_n are written in column form

$$\Phi_n = \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}.$$

2. We shall now consider a more specialized version of a dimer. Suppose each chromophore is diatomic and therefore possesses a single vibrational mode. The molecule of 2:2:1 bicycloheptadiene may serve as a good example of such a dimer. The co-ordinate Q will denote in each chromophore respectively the displacement from the equilibrium for the ground electronic state. Using the harmonic approximation we have for moiety A

$$\begin{cases} V_N = \dfrac{1}{2} k Q_A^2 \\ V_M = V_0 + l Q_A + k' Q_A^2, \end{cases} \tag{8}$$

where V_0 is the "vertical" electronic excitation energy. We shall allow for different equilibrium distance in the excited state. Similarly, we have for $v(L)$:

$$v(L) = v_0 + v_1 L + \frac{1}{2} v_2 L^2 + \ldots \tag{9}$$

and v_0 is by far the most important term and also

$$h'_L - T_L = u_0 + u_1 L + \frac{1}{2}\tilde{k} L^2. \tag{10}$$

Denote by m the effective masses of both Q_A and Q_B modes, and by M that of the L mode.

Now let us make the following co-ordinate transformation:

$$\begin{cases} q_1 = 2^{-1/2}(Q_A + Q_B) \\ q_2 = 2^{-1/2}(Q_A - Q_B). \end{cases} \tag{11}$$

Under this transformation, the Hamiltonian (7) changes into

$$\bar{h} = W_0 + h_1 + h_2 + h_L + \Delta V \sigma_3 + v\sigma_1, \tag{12}$$

where

$$W_0 = V_0 + u_0$$

$$\text{(13)} \quad \begin{cases} \bar{h}_1 = \dfrac{p_1^2}{2m} + 2^{-1/2} l q_1 + \dfrac{1}{4}(k+k')\, q_1^2 \\ \bar{h}_2 = \dfrac{p_2^2}{2m} + \dfrac{1}{4}(k'+k)\, q_2^2 \\ \bar{h}_L = \dfrac{P^2}{2M} + u_1 L + \dfrac{1}{2}\tilde{k} L^2 \\ \Delta V = \dfrac{1}{2}\left[\sqrt{2}\, l + (k'+k)\, q_1\right] q_2 . \end{cases}$$

In these equations p_1, p_2, P denote the momenta canonically conjugate to q_1, q_2, L, respectively. We neglect further the L-dependent terms in v and $\frac{1}{2}(k'_s - k)\, q_1\, q_2$ term in ΔV, thus replacing v by v_0 and ΔV by $2^{-1/2} l q_2$. In actual applications, the terms omitted are often of little account; on the other hand, our coupling problem is complicated enough without them. Because ΔV and v are now independent of q_1 and L, we see that these modes are separable.

Let us now apply the transformation (11) to the Hamiltonian

$$\text{(14)} \qquad h = h_A + h_B + h_L ,$$

which describes the vibrational motions in the ground state. Now, owing to the harmonic approximation,

$$\text{(15)} \quad \begin{cases} V_N(Q_A) = \dfrac{1}{2} k Q_A^2 , \\ V_N(Q_B) = \dfrac{1}{2} k Q_B^2 , \\ V_0(L) = \dfrac{1}{2} K L^2 . \end{cases}$$

The Hamiltonian (14) changes therefore under the transformation (11) to

$$\text{(16)} \qquad h = h_1 + h_2 + h_L ,$$

where

$$\text{(17)} \quad \begin{cases} h_1 = \dfrac{p_1^2}{2m} + \dfrac{1}{2} k q_1^2 \\ h_2 = \dfrac{p_2^2}{2m} + \dfrac{1}{2} k q_2^2 \\ h_L = \dfrac{P^2}{2M} + \dfrac{1}{2} K L^2 . \end{cases}$$

It can easily be seen from (17), that this system is separable in all three modes. Owing to the scalar character of operator h, there is in the ground state no degree of freedom associated with the "spin".

Now let us compare modes corresponding to q_1 and L in the ground and excited states. In our example q_1 refers to the vibrations of both $C = C$ bonds in phase (Fig. 1), and $-L$ to the relative ("flapping") motions of two moieties. Both q_1 and L modes are totally symmetric. Looking at h_1, h_L and $\bar{h}_1$, $\bar{h}_L$ in (17) and (13) we see that the spectrum will be normal with respect to these modes. The progressions in both modes will appear, relative intensities will be determined by the usual Franck-Condon considerations.

The q_1 mode does not couple at all with the "spin" of the excited state. The asymmetric stretching mode q_2 which in our example refers to $C = C$ vibrations

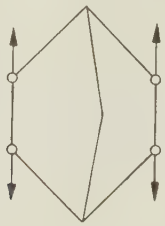

Fig. 1

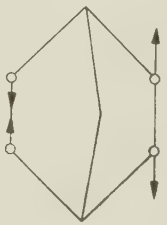

Fig. 2

with a phase difference (Fig. 2) may, however, couple strongly with the "spin" of the excited state, thus showing an exceptional behaviour. Now let us reduce further the Hamiltonian (12) separating out q_1, L modes:

$$(18) \qquad (\bar{h} - W_0 - \bar{h}_1 - \bar{h}_L) = \frac{p_2^2}{2m} + \frac{1}{4}(k'+k)\, q_2^2 + 2^{-1/2} l q_2 \sigma_3 + v_0 \sigma_1 .$$

To simplify notation let us denote by ω the natural frequency, defined as

$$(19) \qquad m\omega^2 = \frac{1}{2}(k+k'),$$

and use the units of action and obvious scaling factor; then we have the following notation

$$(20) \qquad \begin{cases} \omega\eta = \bar{h} - W_0 - \bar{h}_1 - h_L \\ p = p_2 (m\omega)^{-1/2} \\ q = q_2 (m\omega)^{1/2} \\ b = l(2m\omega^3)^{-1/2} \\ C = \dfrac{v_0}{\omega}. \end{cases}$$

In this notation we obtain for our dimer the following final form of the fundamental Hamiltonian, describing vibronic states:

$$(21) \qquad = \frac{1}{2}(p^2+q^2) + bq\sigma_3 + C\sigma_1 .$$

3. The potential energy in the Hamiltonian (21) is:

$$V = \frac{1}{2} q^2 + bq\sigma_3 + C\sigma_1 . \tag{22}$$

Taking into account the familiar properties of σ matrices we can say that motions are to occur on the following potential energy curves:

$$V = \frac{1}{2} q^2 \pm \sqrt{b^2 q^2 + C^2} . \tag{23}$$

It is easily seen from the character of the above potential energy curves that the motion occurring on such sheets will be in general very complex. We may anticipate that only certain limiting cases of motion on those sheets are easy to discuss. Let us remark that in our units the zero point oscillation energy with the Hamiltonian $\frac{1}{2}(p^2+q^2)$ is simply $\frac{1}{2}\hbar$. Now for $|b^2 C| \gg \hbar^2$ or $|b^2 C| \ll \hbar^2$, we will have cases relatively easy to discuss. Indeed, when $|b^2 C| \gg \hbar^2$, the motion is essentially classical, since the separation between the pair of potential energy curves (23) is large as compared with $\hbar$. When $|b^2 C| \ll \hbar^2$, then the terms $bq\,\sigma_3$ and $c\,\sigma_1$ have small effect, compared with the rest of the Hamiltonian and can be treated by the perturbation technique.

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REFERENCES

[1] A. Witkowski and W. Moffitt, J. Chem. Phys., **33** (1960), 872.

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Electronic Spectra of Dimers. II. Perturbation Solutions of the Fundamental Vibronic Equation

by
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Presented by J. WEYSSENHOFF on December 8, 1960

1. It has been shown in [1] and [2] that for a dimer composed of 2 diatomic chromophores (2:2:1 bicycloheptadiene) the asymmetric stretching mode q_2 ($q_2 = 2^{-\frac{1}{2}}(Q_A - Q_B)$ where Q_A, Q_B denote nuclear co-ordinates in diatomic chromophores A, B which refers to $C = C$ vibrations in chromophore with a phase difference, may couple strongly with the electronic motions in the excited electronic state. The reduced Hamiltonian η describing vibronic states of dimer has the following final form (after separating out the vibrational modes which do not couple with electronic motions):

$$\eta = \frac{1}{2}(p^2 + q^2) + bq\sigma_3 + C\sigma_1. \tag{1}$$

In this equation, q denotes the co-ordinate corresponding to the asymmetric stretching mode, p — the canonically conjugate momentum, b gives the change of equilibrium positions for nuclear motions between the ground and excited electronic states, C — the measure of resonance coupling between the moieties of the dimer, all in an appropriate scale, σ_1, σ_3 denote the familiar Pauli matrices. Denoting by k and k' the force constants for nuclei in ground and excited states and by m — the effective mass connected with both Q_A and Q_B modes, we define the natural frequency ω by

$$m\omega^2 = \frac{1}{2}(k' + k).$$

In this notation we have — for the symbols appearing in (1) — the following form:

$$p = p_2 (m\omega)^{-1/2}$$
$$q = q_2 (m\omega)^{1/2}$$
$$b = l (2 m\omega^3)^{-1/2}$$
$$C = \frac{v_0}{\omega},$$

where l is the coefficient of the linear term in the expansion describing the potential energy in the excited state and v_0 is defined as follows: denote by $\Psi_N(q_A, Q_A)$ ground state wave function depending on electronic co-ordinates q_A and parametrically on nuclear co-ordinates Q_A, similarly by Ψ_M the excited state wave function; denote also

$$\pi_A = \psi_M(q_A, Q_A)\,\psi_N(q_B, Q_B); \quad \pi_B = \psi_M(q_B, Q_B)\,\psi_N(q_A, Q_A);$$
$$\pi_0 = \psi_N(q_A, Q_A)\,\psi_N(q_B, Q_B)$$

and by V_{AB} the interaction potential energy between moieties A, B.

Then v_0 is supposed to be constant equal to

$$v_0 = \int_{-\infty}^{+\infty} \pi_A V_{AB}\, \pi_B\, dq_A\, dq_B .$$

Eq. (1) was derived using the electronic basis π_A, π_B Sometimes it is more convenient to use basis $\pi_x = \frac{1}{\sqrt{2}}(\pi_A + \pi_B)$, $\pi_u = \frac{1}{\sqrt{2}}(\pi_A - \pi_B)$. Then (1) takes the form:

$$\eta' = \frac{1}{2}(p^2 + q^2) + bq\sigma_1 + C\sigma_3 . \tag{3}$$

2. The difficulty which arises when trying to solve the eigenvalue problem with the Hamiltonian (3), is easily seen to be connected with the non commutativity of σ_1 and σ_3. We may hope to solve the problems, when either of the coefficients of σ_1 and σ_3 is small compared with the other. Consider first the strong coupling case, when the effect of the change in equilibrium positions is small as compared with the influence of reasonance coupling or $b^2 \ll |C|$. When the term $bq\,\sigma_1$ of η' may be regarded as small, $b^2 \ll |C|$, we can treat it as perturbing the motion described by the Hamiltonian $\frac{1}{2}(p^2+q^2)+C\,\sigma_3$. The energy levels are best established by making the following unitary transformation on the Hamiltonian (3)

$$\bar{\eta} = U^+ \eta'\, U$$

and

$$U = \exp(ibS) \tag{4}$$

with

$$S = (\hbar^2 - 4\,C^2)^{-1}(\hbar p\,\sigma_1 + 2\,Cq\,\sigma_2). \tag{5}$$

By this way we find, that in conformity with the second order in b,

$$\bar{\eta} = \frac{1}{2}(p^2 + q^2) + C\sigma_3 + b^2(4\,C^2 - \hbar^2)^{-1}(2\,Cq^2\,\sigma_3 + \frac{1}{2}\hbar^2). \tag{6}$$

This result shows the spectrum to be composed of levels of the two separate electronic states, each with its own force constant:

$$\eta_+ = \frac{1}{2}(p^2 + f_+ q^2) + e_+ , \qquad \eta_- = \frac{1}{2}(p^2 + f_- q^2) + e_- , \tag{7}$$

where

$$(8) \qquad e_{\pm} = \pm C + \frac{1}{2}\hbar^2 b^2 (4C^2 - \hbar^2)^{-1}, \qquad f_{\pm} = 1 \pm 4Cb^2 (4C^2 - \hbar^2)^{-1}.$$

3. Consider now the weak coupling limit, i.e. the case when term $C\sigma_1$ in the Hamiltonian (1) is small, and may be regarded as perturbing the motion described by the Hamiltonian $\frac{1}{2}(p^2 + q^2) + bq\,\sigma_3$.

We perform the following unitary transformation of the Hamiltonian (1):

$$\eta'' = V^{+} \eta V,$$

where

$$(9) \qquad V = \exp\left(\frac{ibp\sigma_3}{\hbar}\right).$$

Now, since

$$e^{-i\lambda S} A e^{i\lambda S} = A + i\lambda [A, S] + \frac{(i\lambda)^2}{2!} [[A, S], S] + \ldots .$$

where $[A, S]$ denotes the commutator, we obtain:

$$(10) \qquad \eta'' = \frac{1}{2}(p^2 + q^2) - \frac{1}{2} b^2 + C\left[\sigma_1 \cos\left(\frac{2bp}{\hbar}\right) + \sigma_2 \sin\left(\frac{2bp}{\hbar}\right)\right].$$

Choosing the representation in which $\frac{1}{2}(p^2 + q^2)$ and σ_1 are diagonal, and using exponentials, we have:

$$\tilde{\eta} = \frac{1}{2}\left[p^2 + q^2 + C\sigma_3 \left(e^{\frac{2bpi}{\hbar}} + e^{\frac{-2bpi}{\hbar}}\right) - iC\sigma_2 \left(e^{\frac{2bpi}{\hbar}} - e^{\frac{-2bpi}{\hbar}}\right)\right].$$

It is seen that the term with σ_3 of the above equation contributes to the first order, while the term with σ_2 appears only in the second and higher order corrections. The first order energy is easily obtained, after calculation of the integrals of the type

$$\int_{-\infty}^{+\infty} \Phi_n (q - b)\, \Phi_n (q + b)\, dq,$$

where Φ_n are harmonic oscillator wave functions. This integral may be calculated by using in the standard way the generating function for Hermite polynomials.

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REFERENCES

[1] A. Witkowski and W. Moffitt, J. Chem. Phys., **33** (1960), 872.

[2] A. Witkowski, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **9** (1961). 179 [this issue].

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Electronic Spectra of Dimers. III. Intensities

by
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In order to obtain a convenient characteristic of the spectra of dimers, it is desirable to determine the total intensity of the system, the location of the centre of gravity of the spectrum, and its width. We use here, without further explanation, the notation described in the first two papers of this series [1].

The dipole strength for the vibronic transition from the vibrationally unexcited ground-state to the M-th level of the upper state of dimer

$$\Psi_0 \to \Psi_M,$$

where

$$\Psi_0 = \pi_0 \Phi_0$$

$$\Psi_M = \pi_A \alpha_M + \pi_B \beta_M$$

is given by the expression

$$D_{0M} = |\{\{\Psi_0, \vec{\mu}\, \Psi_M\}\}|^2 \tag{1}$$

where $\{\{,\}\}$ denotes integration over electronic and nuclear co-ordinates.

Denote

$$\vec{\mu}_A^{\circ} = \{\pi_0, \vec{\mu}_A \pi_A\}, \quad D = \vec{\mu}_A^{\circ} \cdot \vec{\mu}_A^{\circ}, \quad D\cos\Theta = \vec{\mu}_A^{\circ} \cdot \vec{\mu}_B^{\circ},$$

where $\{,\}$ denotes integration over electronic co-ordinates; then (1) can be written in the following form:

$$\begin{cases} D_{0M} = [(\Phi_0, \alpha_M)(\alpha_M, \Phi_0) + (\Phi_0, \beta_M)(\beta_M, \Phi_0)]\, D + \\ + [(\Phi_0, \alpha_M)(\beta_M, \Phi_0) + (\Phi_0, \beta_M)(\alpha_M, \Phi_0)]\, D\cos\Theta. \end{cases} \tag{2}$$

It is convenient now to introduce the column spinor notation. We denote

$$\varphi_0 = \begin{pmatrix} \Phi_0 \\ 0 \end{pmatrix} \cdot \qquad \varphi_M = \begin{pmatrix} \alpha_M \\ \beta_M \end{pmatrix}$$

and

$$(\Phi_0, \alpha_M) = \langle \varphi_0, \alpha_M \rangle = \langle 0|M\rangle,$$

where the symbol $\langle , \rangle$ denotes integration over vibrational co-ordinates, as well as summation over spinor indices. Using the notation, we can rewrite Eq. (2) in the following form:

$$(3) \qquad \begin{cases} D_{0M} = [\langle 0|M\rangle \langle M|0\rangle + \langle 0|\sigma_1|M\rangle \langle M|\sigma_1|0\rangle] D + \\ + [\langle 0|M\rangle \langle M|\sigma_1|0\rangle + \langle 0|\sigma_1|M\rangle \langle M|0\rangle] D \cos \Theta. \end{cases}$$

If we carry out the summation over all possible transitions from the vibronically unexcited ground state to the vibronic upper states, then we obtain applying the obvious properties of Pauli σ matrices the following result:

$$\sum_M D_{0M} = [\langle 0|0\rangle + \langle 0|\sigma_1^2|0\rangle] D + [\langle 0|\sigma_1|0\rangle + \langle 0|\sigma_1|0\rangle] D \cos \Theta = 2 D.$$

To get a further characteristic of the spectrum, let us calculate its successive moments. Denoting by r the vertical excitation energy, we notice that in our notation:

$$V' = r + bq\, \sigma_3 + C\, \sigma_1$$

$$H \varphi_0 = (T+V) \varphi_0 = E_0 \varphi_0$$

$$H' \varphi_M = (T+V+V') \varphi_M = E_M \varphi_M$$

$$(E_M - E_0)^k = \sum_{M'} \langle M|(H' - E_0)^k|M'\rangle \delta_{MM'}.$$

By this definition, the successive moments M_k are:

$$(4) \qquad M_k = \frac{\sum_M D_{0M} (E_M - E_0)^k}{\sum_M D_{0M}} = (2D)^{-1} \sum_M D_{0M} (E_M - E_0)^k.$$

If we now notice that

$$\sum_M |M\rangle (E_M - E_0)^k \langle M| = \sum_M \sum_{M'} |M\rangle \langle M|(H' - E_0)^k |M'\rangle \langle M'|$$

we can obtain the following result:

$$(5) \qquad \begin{cases} M_k = \frac{1}{2} \{\langle 0|(H' - E_0)^k| + \sigma_1 (H' - E_0)^k \sigma_1 |0\rangle\} + \\ + \frac{1}{2} \cos \Theta \{\langle 0|(H' - E_0)^k |\sigma_1 + \sigma_1 (H' - E_0)^k |0\rangle\}. \end{cases}$$

In particular we obtain:

$$M_1 = r + C \cos \Theta$$

$$M_2 = r^2 + b^2 \langle 0|q^2|0\rangle + C^2 + 2 Cr \cos \Theta.$$

Thus, we have obtained the following results: the total intensity of dimer, D_0, is equal to

$$(6) \qquad D_0 = \sum_M D_{0M} = 2D,$$

The centre of gravity ω_0 of the spectrum is located by the $\omega_0 = M_1/\hbar$, thus (in angular frequency scale) we obtain:

$$\omega_0 = \hbar^{-1}(r + C\cos\Theta). \tag{7}$$

The width of the spectrum, is defined by the equation:

$$\Delta\omega = \hbar^{-1}[M_2 - (M_1)^2]^{1/2}. \tag{8}$$

Thus for our system the width is equal to

$$\Delta\omega = \hbar^{-1}[b^2\langle 0|q^2|0\rangle + C^2\sin^2\Theta]^{1/2}. \tag{9}$$

Let us finally compare the obtained characteristic of the spectrum for a dimer with the analogous characteristic of the spectrum of a monomeric unit of the dimer. For the monomer (denoted, say, by A) has been obtained previously [2] the following result:

$$\begin{cases} D_0 = D \\ M_1 = V_0 + \frac{1}{2}(k' - k)\langle 0|Q_A^2|0\rangle \\ M_2 - (M_1)^2 = l^2\langle 0|Q_A^2|0\rangle + \frac{1}{4}(k' - k)[\langle 0|Q_A^4|0\rangle - \langle 0|Q_A^2|0\rangle^2], \end{cases} \tag{10}$$

where V_0 denotes the vertical excitation energy and the angular brackets denote the expectation values over the ground state. Now we neglect the difference between the force constants for ground (K) state and excited (K') state as we did when deriving the Hamiltonian for the dimers. We can see from (10), that, unless the zero-point amplitudes are unusually large, the second terms in (10) will be very small.

We see from the above equations that the resonance coupling parameter C is the leading factor which is responsible for the change in shape of spectra of dimers, as compared with those of monomers.

A fuller report on the spectra of dimers will appear in Roczniki Chemii. This series of papers owes much to the late professor W. E. Moffitt. The numerous discussions held with him and general stimulation obtained during the author's association with him at Harvard greatly helped in the investigations of the above problems.

The author also wishes to express his gratitude to Dr M. Gouterman for valuable discussions concerning dimers.

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REFERENCES

[1] A. Witkowski, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **9** (1961), 179 and 185.
[2] A. Witkowski and W. Moffitt, J. Chem. Phys., **33** (1960), 879.

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On the Mechanism of Impurity Conduction in Semiconductors I. The Activation Energy of Conductivity

by

JERZY MYCIELSKI

Presented by L. INFELD on January 13, 1961

In 1955 H. Fritzsche observed [1] a new low temperature anomaly in the temperature dependence of conductivity and Hall coefficient in p-type germanium at temperatures between 3°K and 10°K. The impurity (Ga) concentrations ranged from 2.1×10^{16} to 7.3×10^{16} cm.$^{-3}$ and compensation K was about 10%. At temperatures above 10°K a normal extrinsic conduction occurred; at temperatures lower than 3°K impurity conduction with activation energy ϵ_3 about 1×10^{-3} eV was observed. This latter impurity conduction was investigated by many authors and was explained (see e.g. [2]—[4]) as the result of tunnelling of carriers from the ground states of occupied impurity centres to the ground states of unoccupied ones. Such unoccupied majority centres are always present because of compensation.

Similarly as in the extrinsic and the "ϵ_3 range", also in the range of the new anomaly the conductivity depends exponentially on $1/T$. But the important feature is a strong dependence of the activation energy ϵ_2 on the concentration of majority impurities. The activation energy decreases with increasing impurity concentration, changing from 4.04×10^{-3} to 0.93×10^{-3} eV in the impurity concentration range specified above.

The same anomaly occurs also in n-type germanium with impurity (Sb) concentrations from 2.4×10^{16} to 5.5×10^{16} cm.$^{-3}$ and compensation of the order of a few percent [5]. As for p-type — the anomaly was observed in the temperature range 3°K—10°K; in the given range of impurity concentration ϵ_2 decreases from 5.3×10^{-3} to 1.6×10^{-3} eV.

James and Anderson [6] proposed for the interpretation of the "ϵ_2 anomaly" a tunnelling mechanism of conduction along chains formed by the most closely spaced impurities. It is not clear, however, how one can obtain in this way the high values of the activation energy ϵ_2 observed experimentally.

It is also difficult to interpret quantitatively the "ϵ_2 anomaly" conduction as resulting from carrier jumps from the occupied impurity centres to the empty ones through a certain quantized excited state. The system formed by two impurity ions

and one carrier is analogous to the H_2^+ ion. In this ion [7] the energy gap between very closely spaced states $1s\sigma_g$ and $2p\sigma_u$ and other excited states is almost constant and about one half of the isolated centre ionization energy E_a, if the distance between the ions ranges from 4 to 7 effective Bohr radii. This range of distances corresponds to the observed range of impurity concentrations in germanium. Since the ionization energy of Ga in Ge is 1.08×10^{-2} eV [8] and of Sb in Ge — 9.8×10^{-3} eV [9], it seems apparently impossible to obtain in this way the right value and concentration dependence of ϵ_2. It is well known [10], [8], that the shallow impurity centres in *p*-type Ge do not behave strictly as hydrogen-like atoms, but the deviations in the energy gaps from the simple hydrogen-like model do not exceed 25%. They cannot give any substantial improvement. A similar situation occurs in the case of *n*-type Ge [11], [9], with the only distinction that the Sb centre in Ge possesses an excited state very close to the ground state; the energy gap between them is of the order of 10^{-4} eV [12], [13]. This is one order of magnitude smaller than the observed value of ϵ_2.

The question arises, whether the two-centre model is correct. Can "ϵ_2 conduction" be the conduction in an excited impurity band formed by interacting excited states of impurity centres? The carrier can be excited to such a band from the ground state of the centre by phonons. S. Koshino [14] interpreted in this way one pair of curves of ϱ vs. 1/T and R vs. 1/T given in [1]. This result cannot, however, be regarded as of great importance, because the agreement with experimental data was obtained by an arbitrary choice of the values of as many as 9 parameters. In this interpretation ϵ_2 corresponds to the energy gap between the ground state of the impurity centre and the bottom of the excited impurity band, and not to one half of this energy, because owing to compensation, the impurity centres are partially ionized and so, at low temperatures, the Fermi level is situated very near the ground state. However, as follows from the theoretical considerations of W. Baltensperger [15], this energy gap is, independently of impurity concentration, never smaller than 0.6 E_a, which is in disagreement with the observed values of ϵ_2. If one takes into account the statistical distribution of the positions of the impurities in the crystal [16], then the bottom and the top of the impurity band disappear. The density of states of this band has "tails" to lower and higher energies. However, it seems unlikely, that this statistical effect be large enough to explain the right value of ϵ_2 on this way.

Because of all these difficulties it may be interesting to consider another possible mechanism of conduction. The carrier can jump over the Coulomb potential wall from the occupied impurity centre to the empty one. The possibility of such a conduction arises from the fact, that in consequence of carrier-phonon interaction the wave functions of the carriers loose partially their coherence, the energies of the carriers becoming different from the quantized energies of the two-centre potential. We shall consider the case of small compensation ($K \ll 1$).

The possibility of obtaining the activation energy of conductivity ϵ_2 in the proposed mechanism can be seen from Fig. 1. The solid line shows the potential energy of the carrier in the electrostatic field of two impurity ions separated by

distance a. The energy of the carrier in the ground state of the impurity centre, taking into account the interaction with the ionized centre, is

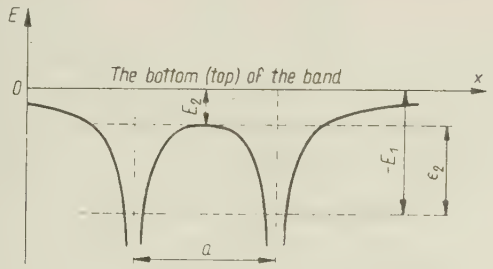

Fig. 1

(1) $$E_1 = -E_a - \frac{e^2}{\varepsilon a}.$$

Here E_a is the ionization energy of an isolated impurity centre, e — the elementary charge, and ε — dielectric constant. The energy gap between the top of the potential wall and the bottom (top) of the conduction (valence) band is

(2) $$E_2 = \frac{4e^2}{\varepsilon a}.$$

The activation energy of conductivity ϵ_2 equals $-E_1 - E_2$, thus

(3) $$\epsilon_2 = E_a - \frac{3e^2}{\varepsilon a}.$$

For simplicity we assume in the following that the majority impurity centres form in the crystal an "impurity sublattice". This assumption gives reasonable results in the theory of the "ϵ_3 anomaly" [3]. We are interested only in the jumps to the nearest neighbours in this sublattice, because, as follows from the formula (3), ϵ_2 rises with rising a and the jumps to larger distances would be much more improbable at low temperatures. For any type of the sublattice we have

(4) $$\frac{1}{a} = SN^{1/3},$$

where S is a parameter depending on the type of sublattice, but always not far from unity; $S = 0.89$ for the face-centred cubic and the hexagonal closed-packed sublattices, $S = 0.92$ for the body-centred cubic sublattice, $S = 1.00$ for the simple cubic sublattice, $S = 1.15$ for the diamond-type sublattice. N is the concentration of the majority impurities, and a denotes now the distance between the nearest neighbours in the impurity sublattice.

From the formulae (3) and (4) we obtain

$$\epsilon_2 = E_a - \frac{3e^2}{\varepsilon} S N^{1/3}. \tag{5}$$

Thus we are led to expect that on the ϵ_2 vs. $N^{1/3}$ plot the experimental points lie on a straight line and that the coefficient S determined from the slope of this line and from formula (5) is close to unity.

Expressing in formula (5) the energies in eV we have

$$\epsilon_2 = E_a - 4.31 \times 10^{-7}\, \varepsilon^{-1}\, S N^{1/3}. \tag{5'}$$

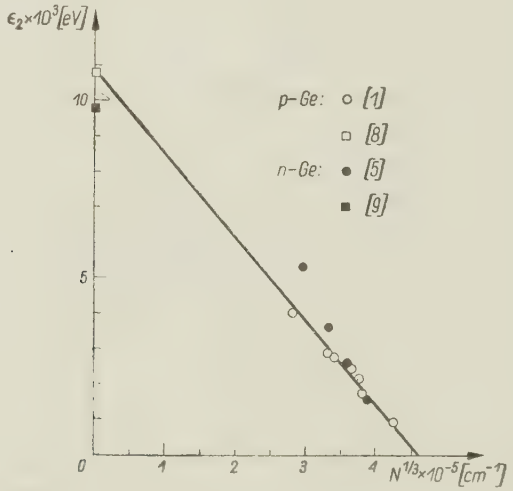

Fig. 2

In Fig. 2 we have the experimental points obtained by H. Fritzsche [1], [5], and the ionization energies of Ga and Sb in germanium [8], [9]. For p-type Ge these points lie well on a straight line. From the slope of this line, taking $\varepsilon = 16.0$ [17], we have

$$S = 0.88. \tag{6}$$

For n-type Ge the points lie also on a straight line with a larger slope, but the line does not coincide with the point given by the ionization energy of Sb in Ge. Perhaps the values of ϵ_2 for n-type Ge were measured with much less accuracy than for p-type; the straight lines on the ϱ vs. $1/T$ plots are much shorter than in the case of the p-type.

In the preceding considerations we neglected the effect of the Coulomb field of the compensating impurities. These impurities, being all ionized, attract the "empty states" of the majority centres. The empty states are then partially bound to the compensating centres; such states do not contribute to the conduction. However, the binding energy is small and for $K \ll 1$ and in the temperature range

considered most of the empty states are not bound [18]. Thus the neglect of this effect is justified.

We would like to point out that formula (5) holds not only for the shallow but also for the deep impurity states and is independent of the band structure. The only assumption concerning the bulk material is the isotropy of the dielectric constant.

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REFERENCES

[1] H. Fritzsche, Phys. Rev., **99** (1955), 406.
[2] H. Fritzsche, M. Cuevas, Phys. Rev., **119** (1960), 1238.
[3] P. Csavinszky, Phys. Rev., **119** (1960), 1605.
[4] Y. Toyozawa, Progr. Theor. Phys., **23** (1960), 378, 380.
[5] H. Fritzsche, J. Phys. Chem. Solids, **6** (1958), 69.
[6] H. M. James, P. W. Anderson, in the Report on the Second Symposium on the Physics of Semiconductors, J. Phys. Chem. Solids, **2** (1957), 76.
[7] D. R. Bates, K. Ledsham, A. L. Steward, Phil. Trans. Roy. Soc. London, A **246** (1953), 215.
[8] P. Fisher, H. Y. Fan, Phys. Rev. Lett., **2** (1959), 456.
[9] H. Y. Fan, P. Fisher, J. Phys. Chem. Solids, **8** (1959), 270.
[10] D. Schechter, Carnegie Inst. of Techn., thesis (1958).
[11] W. Kohn, Solid State Phys., **5** (1957), 257.
[12] E. M. Conwell, Phys. Rev., **99** (1955), 1195;
[13] H. Fritzsche, Phys. Rev., **115** (1959), 336.
[14] S. Koshino, J. Phys. Soc. Japan, **11** (1956), 608.
[15] W. Baltensperger, Phil. Mag., **44** (1953), 1355.
[16] P. Aigrain, Physica, **20** (1954), 978.
[17] F. A. D'Altroy, H. Y. Fan, Phys. Rev., **103** (1956), 1671.
[18] S. H. Koenig, G. R. Gunther-Mohr, [appendix by P. J. Price], J. Phys. Chem. Solids, **2** (1957), 268.

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On the Mechanism of Impurity Conduction in Semiconductors II. The Conductivity in the Case of Strong Carrier-Phonon Interaction

by

JERZY MYCIELSKI

Presented by L. INFELD on January 13, 1961

In the preceding paper [1] we proposed a new mechanism of impurity conduction in semiconductors at low temperatures and calculated the activation energy of conductivity given by this mechanism. The conductivity arises from the carrier jumps over the Coulomb potential wall from the occupied impurity centres to the empty ones. The empty majority centres are present because of compensation. In the following we are interested in the value of the conductivity itself.

If in the proposed mechanism the carrier jumps from the neutral impurity centre to the ionized one, it must firstly be excited to an energy higher than the top of the potential wall and secondly diffuse to the empty centre. Whether the first or the second process limits the conductivity depends on the value of the carrier-phonon interaction. No other interaction can play a role in the carrier's scattering on its way between the two centres. We shall now discuss the case of strong carrier-phonon interaction, in which diffusion is the limiting factor. The probability of excitation of the carrier is so large that the volume density of carriers in the neutral centre is given by the density of thermal equilibrium. In the empty centre the density is zero. Of course, by "density of carriers" we mean the probability of finding the carrier in the unit volume. We shall consider the diffusion current from the neutral centre to the ionized one and obtain first the jumping frequency and then the electric conductivity. We shall be interested in the case of small compensation ($K \ll 1$).

The conductivity obtained in this way is expected to be equal to the observed one in cases in which the carrier-phonon interaction is strong, and to be larger than the experimental values, if this interaction is weak. In other words, if the proposed mechanism determines the conduction, our calculated conductivity must not be smaller than the observed one.

Jumping frequency

We first have to obtain the density of carriers in the neutral centre at the thermal equilibrium. We assume that the strong carrier-phonon interaction is weak enough to permit the treatment of the ground state of the carrier at the impurity centre

as a stationary quantized state. We are interested in the density at such a distance from the ion, at which the density of the ground state can be neglected (e.g. two or three effective Bohr radii). The field of the ion at this distance can be treated as a Coulomb field; very close to the ion there are deviations depending on the nature of the impurity.

Let us introduce the co-ordinate system x, y, z in such a way, that the ion of the neutral centre have the co-ordinates $(-a/2, 0, 0)$ and that of the ionized one — $(a/2, 0, 0)$. We denote by q the electric charge of the carrier ($|q| = e$) and by φ the potential resulting from the two centres (ε being dielectric constant)

$$\varphi = -\frac{q}{\varepsilon}\left[\frac{1}{\sqrt{\left(x+\frac{a}{2}\right)^2+y^2+z^2}}+\frac{1}{\sqrt{\left(x-\frac{a}{2}\right)^2+y^2+z^2}}\right]. \tag{1}$$

By n_{rr} we denote the density of carriers at thermal equilibrium. If, as was assumed, the carrier-phonon interaction is strong, then the excited quantized states of the two-centre potential disappear and the well-known formula holds

$$n_{rr} = \frac{1}{\sqrt{2}\,\hbar^3}\left(\frac{m^* kT}{\pi}\right)^{3/2} f. \tag{2}$$

Here m^* is the density of states effective mass of the carrier, and f is the occupation probability of the states at the bottom (top) of the deformed conduction (valence) band, i.e. of the states of energy $q\varphi$.

Because of carrier-carrier interaction it is impossible to have two or more carriers at one centre; the states of the neutral centre are occupied alternatively by one carrier only. This yields

$$\frac{f_{\mathrm{I}}}{f_{\mathrm{II}}} = \exp\frac{E_{\mathrm{II}} - E_{\mathrm{I}}}{kT}, \tag{3}$$

where $f_{\mathrm{I,II}}$ and $E_{\mathrm{I,II}}$ are the occupation probabilities and energies for the states I and II, respectively. Let us take for state I the state at the bottom (top) of the deformed band; then $f_{\mathrm{I}} = f$ and $E_{\mathrm{I}} = q\varphi$. For state II we take one of the ground states in the neutral centre; then at low temperatures we have $f_{\mathrm{II}} \cong 1/\gamma$, where γ is the degeneracy of the ground energy level, and $E_{\mathrm{II}} = E_1$ (see formula (1) in [1]). Hence (E_a being the ionization energy of an isolated centre),

$$f = \frac{1}{\gamma}\exp\left(-\frac{E_a + \frac{e^2}{\varepsilon a} + q\varphi}{kT}\right). \tag{4}$$

We put

$$n_{r0} = \frac{1}{\sqrt{2}\,\hbar^3\gamma}\left(\frac{m^* kT}{\pi}\right)^{3/2}\exp\left(-\frac{E_a + \frac{e^2}{\varepsilon a}}{kT}\right). \tag{5}$$

From formulae (2), (4) and (5) we obtain finally

$$n_{rr} = n_{r0} \exp\left(-\frac{q\varphi}{kT}\right). \tag{6}$$

Now we shall write the diffussion equation for the carrier. We assume that the carrier's diffusion coefficient D and mobility μ in the band, both limited by the carrier-phonon interaction only, are isotropic. We denote by $\vec{J}$ the density of the probability current of the carrier and by n_r the density of carriers. Using the Einstein formula

$$\mu = \frac{e}{kT} D, \tag{7}$$

one obtains

$$\vec{J} = -\frac{kT\mu}{e} \operatorname{grad} n_r - \operatorname{sgn}(q)\, \mu n_r \operatorname{grad} \varphi. \tag{8}$$

Now we impose the boundary conditions: on a surface surrounding the ion of the neutral centre (i.e. the point $(-a/2, 0, 0)$) and not very close to it we put $n_r = n_{rr}$ and on an analogous surface surrounding the ion of the ionized centre (i.e. the point $(a/2, 0, 0)$) we put $n_r = 0$. Outside of these surfaces $\operatorname{div} \vec{J} = 0$; using formula (8) and $\Delta\varphi = 0$ we obtain

$$\Delta n_r + \frac{q}{kT} \operatorname{grad} \varphi \operatorname{grad} n_r = 0. \tag{9}$$

Let us define a new function χ by the equation

$$n_r = n_{r0}\, \chi \exp\left(-\frac{q\varphi}{kT}\right). \tag{10}$$

This yields instead of (9)

$$\Delta\chi - \frac{q}{kT} \operatorname{grad} \varphi \operatorname{grad} \chi = 0. \tag{11}$$

Using (6) we express now the boundary conditions in terms of χ. On the surface surrounding the point $(-a/2, 0, 0)$ $\chi = 1$ and on the surface surrounding the point $(a/2, 0, 0)$ $\chi = 0$.

We now focuse our attention on the vicinity of the point $(0, 0, 0)$. There the potential energy of the carrier has a smaller value than at any point on the $x = 0$ plane. Thus most current flows here. If we neglect the terms of third and higher orders in co-ordinates, then φ in the vicinity of the point $(0, 0, 0)$ is given by the formula

$$\varphi \cong -\frac{4q}{\varepsilon a} + \frac{8q}{\varepsilon a^3}(y^2 + z^2 - 2x^2). \tag{12}$$

With φ of the shape (12) the Eq. (11) has a family of solutions depending on x only; these are the solutions of the equation

$$\chi'' + \frac{32\, e^2}{\varepsilon a^3\, kT} x\chi' = 0. \tag{13}$$

The general solution of this equation is (A, B being arbitrary constants)

$$\chi = A \int_{-\infty}^{x} \exp\left(-\frac{16\,e^2}{\varepsilon a^3 kT}\, t^2\right) dt + B. \tag{14}$$

Our boundary conditions to a good approximation are fulfilled for small values of kT, if for $x \to -\infty$ we have $\chi \to 1$, and for $x \to +\infty$ we have $\chi \to 0$. This indeed holds, if

$$A = -\frac{4\,e}{\sqrt{\pi \varepsilon a^3 kT}}, \tag{15}$$

$$B = 1. \tag{16}$$

We shall write now the x-component of the density of current on the plane $x = 0$. The formulae (8), (10), (5), (12), (14)—(16) yield

$$J_x(x=0) = \frac{2\,k^2 T^2 \mu}{\pi^2 \gamma \hbar^3} \sqrt{\frac{2\,m^{*3}}{\varepsilon a^3}} \exp\left(-\frac{\epsilon_2}{kT}\right) \exp\left[-\frac{8\,e^2}{\varepsilon a^3 kT}(y^2 + z^2)\right], \tag{17}$$

where ϵ_2 is given by the formula (3) in [1].

The total current flow from the neutral centre to the ionized one, i.e. the jumping frequency W is obtained as

$$W = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J_x(x=0)\, dy\, dz. \tag{18}$$

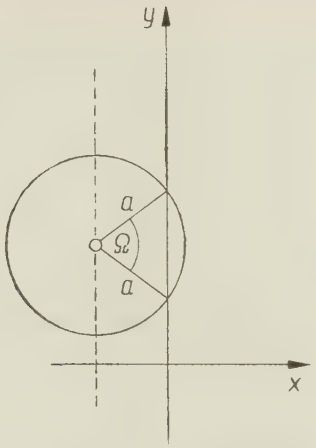
Using (17) this is

$$W = \frac{\sqrt{2m^{*3} \varepsilon a^3 k^3 T^3 \mu}}{4\pi e^2 \hbar^3 \gamma} \exp\left(-\frac{\epsilon_2}{kT}\right). \tag{19}$$

Conductivity

To obtain the electric conductivity σ_2 given by the proposed mechanism we have first to know the diffusion coefficient D_2 for the empty states of impurities. Similarly as in [1], we assume that the majority of impurity centres form in the crystal an "impurity sublattice". P denotes the number of the nearest neighbours, separated by distance a, in this sublattice; $P = 12$ for the face-centred cubic and the hexagonal closed-packed sublattices, $P = 8$ for the body-centred cubic sublattice, $P = 6$ for the simple cubic sublattice, $P = 4$ for the diamond-type sublattice. Let us assume for a moment that in the semiconductor there exists a constant gradient of concentration n of the empty impurity centres in the direction of the x-axis, the concentration of impurity centres being constant. Here we take into account only the "free" empty states of the impurity centres i.e. these not bound to the compensating impurities. The probability that in one second the empty state of the centre will jump the distance a in the direction belonging to the solid angle Ω i.e. the carrier will jump in the opposite direction, is equal to $PW\Omega/4\pi$. This gives the probability of the jump through the $x = 0$ plane for the empty state at the

distance $|x|$ from this plane, if the solid angle Ω corresponds to the surface cut off by the $x = 0$ plane from the sphere of radius a and the centre on the empty impurity (see Figure). Then



$$\Omega = \begin{cases} 2\pi\left(1 - \dfrac{|x|}{a}\right) & \text{for } |x| < a, \\ 0 & \text{for } |x| > a. \end{cases} \tag{20}$$

The jumping frequency through the $x = 0$ plane given by the ionized centres contained in a volume element $dx\ dy\ dz$ is

$$PW \frac{\Omega}{4\pi} n\, dx\, dy\, dz, \tag{21}$$

where

$$n = n(x = 0) + \frac{dn}{dx} x. \tag{22}$$

We obtain the jumping frequency to the right through one cm.^{2} of the $x = 0$ plane using (22) and (20) and integrating (21) over the volume $-\infty < x < 0$, $0 < y < 1$, $0 < z < 1$:

$$\frac{1}{4} PaW \left[n(x = 0) - \frac{a}{3} \frac{dn}{dx} \right]. \tag{23}$$

In a similar way for the jumps to the left we have

$$\frac{1}{4} PaW \left[n(x = 0) + \frac{a}{3} \frac{dn}{dx} \right]. \tag{24}$$

The density of the diffusion current j through the $x = 0$ plane equals the difference of (23) and (24):

$$j = -\frac{Pa^2 W}{6} \frac{dn}{dx}. \tag{25}$$

This yields

$$D_2 = \frac{Pa^2 W}{6}. \tag{26}$$

From the Einstein formula

$$\sigma_2 = \frac{e^2}{kT} D_2 n \tag{27}$$

we obtain

$$\sigma_2 = \frac{e^2 Pa^2 Wn}{6kT}. \tag{28}$$

This formula holds for any jumping mechanism. If the compensation $K \ll 1$ then in our temperature range most of the empty states are free i.e. they are not bound to the compensating impurities [2]. Thus, we can put in (28) $n \cong KN$, N being the concentration of the majority impurities. Using the formulae (19) and (4) in [1] we obtain the conductivity given by the proposed mechanism if the carrier-phonon interaction is strong:

$$\sigma_2 = \frac{\sqrt{2\varepsilon m^{*3}}\, PKk^2 T^2 \mu}{24\pi\hbar^3 \gamma S^{7/2} N^{1/6}} \exp\left(-\frac{\epsilon_2}{kT}\right). \tag{29}$$

Here S is defined by the formula (4) in [1], and ϵ_2 by the formula (5) in [1].

Thus, we have the conductivity with the activation energy ϵ_2. The coefficient before the exponential function, denoted C_2 by H. Fritzsche [3], depends weakly on N and T, because the dependence of μ on T is between $T^{-1.5}$ to $T^{-2.5}$.

If we express σ_2 in ohm^{-1} cm.$^{-1}$ and μ in cm.^{2} V^{-1} sec.$^{-1}$ then the formula (29) has the form (m — free electron mass)

$$\sigma_2 = 2.80\times 10^{-3}\, \varepsilon^{1/2} \left(\frac{m^*}{m}\right)^{3/2} \gamma^{-1} S^{-7/2} PN^{-1/6} KT^2 \mu \exp\left(-\frac{\epsilon_2}{kT}\right). \tag{29'}$$

We shall compare now the coefficient C_2 given by the formula (29′) with the values obtained by H. Fritzsche [3], [4]. Using the subscript h for p-type Ge and e — for n-type Ge, we take

$\varepsilon = 16.0$ (after [5]);
$m_h^*/m = 0.36$ (after [6]);
$m_e^*/m = 0.55$ (after [6]);
$\gamma_h = 4$ (after [7]);
$\gamma_e = 2$ (after [8], [9]);
$S_{h,e} = 0.88$ (from the formula (6) in [1]); for n-Ge a rather arbitrary value);
$P_{h,e} = 12$ (as for the closed-packed structure, because $S_{h,e}$ have a value similar to that characteristic for this structure);
$N_{h,e} = 4.5\times 10^{16}$ cm^{-3} (after [3], [4]);
$K_h = 0.1$ (after [3]);
$K_e = 0.03$ (an approximate value after [4]);
$T = 5$°K (after [3], [4]);

μ_h $= 9.1\times10^8\ T^{-2.3}$ cm.^{2} V^{-1} sec.$^{-1}$ (an empirical formula for the phonon-limited mobility obtained with this coefficient by M. B. Prince [10] for the temperature region above 200°K);

μ_e $= 3.5\times10^7\ T^{-1.6}$ cm.^{2} V^{-1} sec.$^{-1}$ (M. B. Prince [10] — the formula obtained for the temperature region above 220°K).

From formula (29′) we have

$$C_{2h} = 1.0\times10^3 \text{ ohm}^{-1} \text{ cm.}^{-1}, \tag{30}$$

$$C_{2e} = 1.4\times10^2 \text{ ohm}^{-1} \text{ cm.}^{-1}. \tag{31}$$

The experimental values of C_{2h} obtained by H. Fritzsche [3] for *p*-type germanium are, independently of N, about 4.2 ohm^{-1} cm^{-1}, so the theoretical value is about 240 times too high. For *n*-type germanium the plot in paper [4] gives (for the sample "—20") $C_{2e} \cong 6$ ohm^{-1} cm^{-1}, so the theoretical value is here about 23 times too high. Certainly the very far extrapolation consisting of the use of the formulae for μ at the low temperatures results in large errors in the theoretical values. However, the fact that the experimental values of C_2 are much smaller than those given by formula (29) assures, on one hand, the fulfilment of the necessary condition for the possibility of the proposed mechanism, and, on the other hand, is consistent with the weak carrier-phonon interaction in germanium. This weak interaction yields the high mobilities observed. Therefore the limiting process for the conduction is the excitation of carriers from the ground states to energies higher than ϵ_2. This process requires separate consideration.

Formula (29) can have an immediate application, and not only for giving the upper limit of the conductivity, but in the case of a semiconductor with strong carrier-phonon interaction. Unfortunately, up to the present time, we have no experimental results on any such material.

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REFERENCES

[1] Jerzy Mycielski, Bull. Acad. Polon. Sci. Sér. sci. math., astr. et phys. [this issue].,
[2] S. H. Koenig, G. R. Gunther-Mohr (appendix by P. J. Price), J. Phys. Chem. Solids **2** (1957), 268.
[3] H. Fritzsche, Phys. Rev. **99** (1955), 406.
[4] — , J. Phys. Chem. Solids **6** (1958), 69.
[5] F. A. D'Altroy, H. Y. Fan, Phys. Rev. **103** (1956), 1671.
[6] H. Brooks, Adv. in Electronics and Electr. Phys. **7** (1955), 85.
[7] D. Schechter, Carnegie Inst. of Techn. thesis (1958).
[8] E. M. Conwell, Phys. Rev. **99** (1955), 1195.
[9] H. Fritzsche, Phys. Rev. **115** (1959), 336.
[10] M. B. Prince, Phys. Rev. **92** (1953), 681.

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On Some Problems of the *S*-Matrix Formalism for Unstable Particles

by

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Since the energy (or mass) spread of all unstable elementary particles calculated form the formula

$$\Delta E \cdot \Delta \tau \simeq h \tag{1}$$

is very small, reactions between stable as well as unstable elementary particles are usually described by means of the conventional *S*-matrix. In such a treatment we simply neglect the energy spreads and finite lifetimes of unstable particles and ignore the fact that they cannot form asymptotic states in the usual sense of these words. However, in order to describe unstable particles one must introduce into the *S*-matrix some terms which describe decays and are — strictly speaking — not allowed for stable particles of definite mass. This leads, of course, to some inconsistencies in the extended *S*-matrix formalism itself. The aim of this paper is to show that the mass spread can be introduced into the *S*-matrix formalism and that in this way at least some of the inconsistencies will be removed.

As an example let us consider one channel decay of

$$\Sigma^- \to n + \pi^-. \tag{2}$$

We write the *S*-matrix in the form [1], [2]

$$S = 1 + iM + iT + iR, \tag{3}$$

where [2]

$$M = \int \delta_4 (p_1 - p_2 - p_3) \prod_{k=1}^{3} \delta (p_k^2 - m_k^2)\, d_4 p_k \cdot \tag{4}$$

$$\cdot \{\overline{\psi}_2 (a + a' \gamma_5) \psi_1 \varphi_3^+ + \varphi_3 \overline{\psi}_1 (a - a' \gamma_5) \psi_2 e^{i\eta}\}$$

describes all processes connected with (2) by the substitution rule. The indices 1, 2, 3 correspond to Σ^-, n and π^-, respectively. In choosing the form of (3) we have assumed the validity of time reversal invariance and definite masses m_1 and m_3 for Σ^- and π^-, in spite of their finite lifetimes. The matrix T describes $\pi^- - n$

scattering. The matrix R describes all other processes. Neither T nor R give any contribution to the decay matrix elements which in the c.m.s. system have the form

$$(5) \qquad n\pi\,|M|\,\Sigma\rangle = \delta_4(p_1 - p_2 - p_3)\sqrt{\frac{m_2}{2E_2E_3}}\,\bar{u}_2^r(p_2)\,(a + a'\gamma_5)\,u_1^s(0)$$

with $p_1 = (m_1, 0, 0, 0)$; r and s denote polarizations of both fermions. The form (5) of the decay matrix can now be used for studying various properties of the decay.

There is, however, some trouble with the unitarity of the S-matrix, containing terms M of the form (5), and also some difficulties with the interpretation. In fact, from the unitarity condition $SS^+ = 1$ we obtain, after inserting (3) into (5), the following set of interdependent equations, [2]:

$$(6) \qquad i\langle n\pi\,|M - M^+|\,\Sigma\rangle = -\sum_{n''\pi''}\langle n\pi\,|T|\,n''\pi''\rangle\langle n''\pi''|M^+|\Sigma\rangle$$

$$(7) \qquad \begin{cases} i\langle n\pi\,|T - T^+|\,n'\pi'\rangle = -\sum_{n''\pi''}\langle n\pi\,|T|\,n''\pi''\rangle\langle n''\pi''\,|T^+|\,n'\pi'\rangle + \\ -\sum_{\alpha}\langle n\pi\,|M|\,\alpha\rangle\langle\alpha\,|M^+|\,n'\pi'\rangle - \sum_{\beta}\langle n\pi\,|R|\,\beta\rangle\langle\beta\,|R^+|\,n'\pi'\rangle\,. \end{cases}$$

The set of states $|\beta>$ comprises all physical states different from $|n''\pi''>$ and $|\Sigma>$ states which can be reached from the $|n'\pi'>$ without violating any of the conservation laws. If the initial energy is not too high (e.g. smaller than 130 MeV) we can neglect the last term on the right hand side of (7) obtaining in this way two sets of equations involving the matrix elements of M and T only. Next we would be inclined to neglect also the second term in (7), as it is the square of a small matrix., i.e. proportional to the square of the weak coupling constant. After this neglect, Eq. (7) would become completely independent of (6). This means that we take into account the influence of $n - \pi^-$ scattering upon Σ^- decay, but neglect the influence of Σ^- decay upon $n - \pi^-$ scattering, even at this particular energy at which we have to do with some kind of resonance scattering due to the formation of an Σ^- particle. It is easy to check by inserting (5) into (7) that, in fact, the second term in (7) is actually infinite (of course, apart from the usual conservation δ_4). The second trouble is that we are dealing with a phenomenological S-matrix which should not contain any terms leading to infinities. Thus, in order to avoid these difficulties we shall now try to introduce the energy spread of the Σ^- particle into the S-matrix.

The relativistic invariance of the theory will be maintained, if we introduce in (4) an integration over the mass m_1 regarded as a variable parameter

$$(8) \qquad M = \int \delta_4(p_1 - p_2 - p_3)\,dm_1 \prod_{k=1}^{3}(p_k^2 - m_k^2)\,d_4p_k \cdot \\ \cdot \{\bar{\psi}_2(a + a'\gamma_5)\psi_1\varphi_3^+ + \varphi_3\bar{\psi}_1(a - a'\gamma_5)\psi_2 e^{i\eta}\}.$$

The constants a, a' and the fields ψ_1 and $\bar{\psi}_1$ are to be regarded as depending on m_1. Instead of the usual anticommutation relations between annihilation and

creation operators for a particle of definite mass we have now for the operators from the continuous mass spectrum of the Σ^- particle the following relation

$$\{a_r(\vec{p}_1, m_1), a_s^+(\vec{p}_1', m_1')\}_+ = \delta_{rs}\,\delta(m_1 - m_1')\,\delta_3(\vec{p}_1 - \vec{p}_1'). \tag{9}$$

The matrix element which describes the transition of the m_1 component of the Σ^- particle into $|n\pi^-\rangle$ is now in c.m.s.

$$\langle n\pi\, M|\Sigma(m_1)\rangle = \delta_4(p_1(m_1) - p_2 - p_3)\sqrt{\frac{m_2}{2E_2E_3}}\,\bar{u}_2^r(\vec{p}_2)\,(a(m_1) + a'(m_1)\gamma_5)\,u_1^s(0), \tag{10}$$

where in c.m.s. $p_1(m_1) = (m_1, 0, 0, 0)$. In the second term on the right hand side of (7) we must now integrate over the mass m_1 of the intermediate Σ particle. Thus. we obtain a finite contribution from this term, ridding ourselves of one of the main difficulties connected with the introduction of decay terms into the S-matrix.

Let us write the explicit form of Eqs. (6) and (7) obtained after inserting (10). Since we are interested in the resonance region, where, due to the presence of the parity violating the M term, we cannot neglect without investigating parity violation in the scattering, we take the scattering matrix elements in the form

$$\langle n\pi|T|n'\pi'\rangle = \delta_4(p_2 + p_3 - p_2' + p_3')\frac{m_2}{2E_2E_3}\,\bar{u}_2^s(\vec{p}_2)\,[A + B(P\gamma) + C\gamma_5(P\gamma)]\,u_2^r(\vec{p}_2), \tag{11}$$

where the four vectors $P = \frac{1}{2}(p_3 + p'_3)$. A, B and C are scalar functions of two independent scalar products of the momenta, e.g. $p_2 p_3$ and $p_2 p'_3$. The form (11) satisfies time reversal invariance. The term proportional to C which violates parity conservation, is definitely not important outside the resonance region.

It may, however, be of some importance for this region.

With (8), (10) and (11) we obtain

$$i\,\bar{u}_2^r(\vec{p}_2)\,[a - b^* + (a' - b'^*)\gamma_5]\,u_1^s(0) = -\frac{p}{4m_1}\int d\Omega''\,u_2^r(\vec{p}_2)\cdot [A + B(\gamma P) + C\gamma_5(\gamma P)]\,[(\gamma p'' + m_2)(b^* + b'^*\gamma_5)]\,u_1^s(0), \tag{6'}$$

$$i\,\bar{u}_2^r(\vec{p}_2)\,[A - A^* + (B - B^*)(\gamma P) + (C - C^*)\gamma_5(\gamma P)]\,u_2^n(\vec{p}'_2) = -\frac{p}{4m_1}\int d\Omega''\,\bar{u}(\vec{p}_2)\,[A' + B'(\gamma P') + C'\gamma_5(\gamma P')](\gamma p'' m_2)\,[A'' + B''(\gamma P'') + C''\gamma_5(\gamma P'')]\,u_2^s(\vec{p}_2) - \frac{1}{2}\,\bar{u}_2^r(\vec{p}_2)\,(a + a'\gamma_5)\,(1 + \gamma_4)\,(a^* + a'^*\gamma_5)\,u_2^s(\vec{p}'_2), \tag{7'}$$

where $p = |\vec{p}_2|$, $P' = \frac{1}{2}(p_2 + p'')$, $P'' = \frac{1}{2}(p_2' + p'')$; p'' denotes the intermediate neutron momentum and $d\,\Omega''$ — intergration over its directions, $b = a\,e^{i\eta}$ and $b' = a'\,e^{i\eta}$ are constants.

Relation (6) is equivalent to the following set of two equations

$$(12)\quad i(a - b^*) = -\frac{p}{4m_1}\int d\Omega''\,\{b^*\,[A(E_2+m_2) - A(E_2-m_2)\,z + B(m_1-m_2)\times$$
$$\times(E_2+m_2) + B(m_1+m_2)(E_2-m_2)\,z] + b'^*\,C(E_2-m_2)\,m_1(1+z)\},$$

$$(13)\quad i(a' - b'^*) = -\frac{p}{4m_1}\int d\Omega''\,\{b'^*\,[A(m_2-E_2) + A(m_2+E_2)\,z + B(m_2+m_1)\times$$
$$\times(E_2-m_2) + B(m_1-m_2)(E_2+m_2)] + b^*\,Cm_1(E_2+m_2)(1+z)\},$$

where $z = \cos(\vec{p}_2\vec{p}'')$.

If we now neglect the influence of the Σ^* particle formation in the resonance region we can put $C = 0$ and, inserting the well known expression for the functions A, B expressed by the phase shifts, we obtain the well known result

$$(14)\qquad \frac{a'|a'|}{a|a'|} = \exp i(\delta_{p1/2} - \delta_{s1/2}),$$

which has been derived by the use of other methods by Lee and Yang [3] and others. However, having an exact set of Eqs. (6), (7) we can verify to what extent the neglects leading to (14) can be justified.

An exact analysis of (6) and (7) from this point of view will be given in another paper.

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REFERENCES

[1] J. Werle, Nuovo Cim., **9** (1958), 569.
[2] — , ibid., **9** (1958), 1091.
[3] T. D. Lee and C. N. Yang, Phys. Rev., **108** (1957), 1645.

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On the Angular Momentum Weight Factor in the Statistical Theory of Multiple Particle Production. I

by

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1. Introduction

In a recent work [1]*) I proposed a new formulation of the statistical theory of multiple particle production, in which the angular momentum of the system is strictly conserved. In this formalism the relative magnitude of the probability of finding N secondary particles, when the total available energy W and the total angular momentum L (one of its components being M) of the assembly of the secondary particles are specified, is given by the following expression:

$$(1.1)\qquad S_N(W;L,M)=\left(\frac{2}{\pi}\right)^N \sum_{l_1,\ldots,l_N}^{(L)} Z_{N+1}(l_1,\ldots,l_N,L)\cdot\{n_1!\,n_2!\ldots n_\nu!\}^{-1}\times$$

$$\cdot\prod_{i=1}^{N}\left[\int_0^R dr_i\int_0^\infty dp_i\,\mathcal{L}_i\{p_i r_i j_{l_i}(p_i r_i)\}^2\right]\cdot\delta\left(W-\sum_{i=1}^{N}\sqrt{p_i^2+\mu^2}\right).$$

Here $Z_{N+1}(l_1,\ldots,l_N,L)$ denotes the number of independent eigenstates with the eigenvalue L of the total angular momentum, one of its components being fixed, which are involved in the direct product of N one-particle eigenstates of angular momentum, with respective eigenvalues $l_1,\ldots,l_{N-1}$ and l_N. The summation $\sum^L$, which originally meant the combinations of l_i's that lead to the resultant value L, can be extended to all non-negative values of l_i's, if the coefficients Z's are properly defined, so that unnecessary terms will be multiplied by vanishing values of Z's.

*) See Eq. (2.18) of [1]. Here $\sum_{\varrho=1}^{\nu} n_\varrho = N$, and $n_1,\ldots,n_\nu$ denote the numbers of those particles which have identical l's. $\mathcal{L}_i$ represents the effects of relativistic reduction of the interaction. (For instance it can be taken as $\mathcal{L}_i=\mu/\sqrt{p_i^2+\mu^2}$). R is the radius of the interaction region. $j_l(x)=\sqrt{\frac{\pi}{2x}}\,J_{l+\frac{1}{2}}(x)$.

Formula (1.1) has been written for the simplest case when the secondary particles consist of a single kind of bosons. Generalization to more complicated cases is straightforward: If these N secondary particles have spins $\sigma_1, \sigma_2, \ldots, \sigma_N$, respectively, then we have instead of (1.1), *)

$$(1.2) \qquad S_N(W; J, M) = \left(\frac{2}{\pi}\right)^N \sum_{l_1, \ldots, l_N}^{(J)} Z_{2N+1}(l_1, \ldots, l_N, \sigma_1, \ldots, \sigma_N, J) \cdot Q \times$$

$$\times \prod_{i=1}^{N} \left[\int_0^R dr_i \int_0^\infty dp_i \, \mathcal{L}_i \{p_i r_i j_{l_i}(p_i r_i)\}^2 \right] \cdot \delta\left(W - \sum_{i=1}^{N} \sqrt{p_i^2 + \mu^2}\right).$$

The factor Z_{2N+1} reduces to $Z_{N+1}(l_1, \ldots, l_N, L)$ when all the σ_i's are equal to zero (see (2.2) below). The second factor Q represents the effects of Fermi and/or Bose statistics of the secondary particles, and is reduced to $\{n_1! n_2! \ldots n_\nu!\}^{-1}$ in the simplest case of a single kind of bosons. The weight factor due to the isospin conservation should be also included here.

Further we can specify, if we like, a definite parity of the assembly of secondary particles. When $P = \pm 1$ is prescribed, we have to insert into the summand an additional factor:

$$(1.3) \qquad \frac{1}{2}\{1 \pm (-1)^{l_1 + l_2 + \ldots + l_N + N(-)}\},$$

where $N(-)$ denotes the number of secondary particles with odd intrinsic parity.

For the purpose of further application of these formulae to actual problems, I should like to present in this note some general properties and methods of evaluation of these Z-factors. Their asymptotic behaviour for large values of N will be discussed in the second note.

2. Definition and fundamental properties

Let us define our Z_n's by the equation: **)

$$(2.1) \qquad \mathcal{D}^{(l_1)} \times \mathcal{D}^{(l_2)} \times \ldots \times \mathcal{D}^{(l_{n-1})} = \sum_{\lambda} Z_n(l_1, l_2, \ldots, l_{n-1}, \lambda) \cdot \mathcal{D}^{(\lambda)},$$

where $\mathcal{D}^{(m)}$ denotes the $(2m+1)$-dimensional irreducible representation of the three-dimensional rotation group. In other words, $Z_n(l_1, l_2, \ldots, l_{n-1}, l_n)$ means the number of independent eigenstates with total angular momentum l_n, with a certain fixed value of one of its components, which are included in the product of $(n-1)$ one-particle eigenstates with respective angular momentum l_1, l_2, l_{n-2} and l_{n-1}.

In the simplest case of (1.1) all the l_i's are non-negative integers. In order to include more general cases such as (1.2), however, we shall admit that some of the l_i's — necessarily even number of them — in the arguments of Z_n can be half-

*) In (1.2) J is written instead of L, according to the conventional notation.

**) For the moment Z_n's are defined for $n \geq 2$. Z_1 is defined later by (2.6).

odd positive integers. In practical application some of the l_i's are regarded as orbital angular momenta, and the others as spin angular momenta.

For example, we have

$$\mathcal{D}^{(1)} \times \mathcal{D}^{(2)} \times \mathcal{D}^{(2)} = \mathcal{D}^{(0)} + 3\,\mathcal{D}^{(1)} + 3\,\mathcal{D}^{(2)} + 3\,\mathcal{D}^{(3)} + 2\,\mathcal{D}^{(4)} + \mathcal{D}^{(5)},$$

$$\mathcal{D}^{(1/2)} \times \mathcal{D}^{(1/2)} \times \mathcal{D}^{(0)} \times \mathcal{D}^{(1)} = \mathcal{D}^{(0)} + 2\,\mathcal{D}^{(1)} + \mathcal{D}^{(2)};$$

consequently,

$$\begin{aligned} &Z_4(1,2,2,0) = 1, && Z_5(1/2, 1/2, 0, 1, 0) = 1,\\ &Z_4(1,2,2,1) = 3, && Z_5(1/2, 1/2, 0, 1, 1) = 2,\\ &Z_4(1,2,2,2) = 3, && Z_5(1/2, 1/2, 0, 1, 2) = 1,\\ &Z_4(1,2,2,3) = 3, && Z_5(1/2, 1/2, 0, 1, 3) = 0,\\ &Z_4(1,2,2,4) = 2, && \ldots\ldots\ldots\ldots\\ &Z_4(1,2,2,5) = 1,\\ &Z_4(1,2,2,6) = 0,\\ &\ldots\ldots\ldots \end{aligned}$$

By multiplying both sides of (2.1) by $\mathcal{D}^{(l_n)}$, we can easily see that

$$Z_{n+1}(l_1, l_2, \ldots, l_n, 0) = Z_n(l_1, l_2, \ldots, l_n), \tag{2.2}$$

and this affords an alternative physical interpretation: $Z_n(l_1, \ldots, l_{n-1}, l_n)$ represents how many times the eigenstates with vanishing total angular momentum are involved in the direct product of n one-particle eigenstates with angular momenta $l_1, l_2, \ldots, l_{n-1}$ and l_n, respectively. This relation shows that $Z_n(l_1, l_2, \ldots, l_{n-1}, l_n)$ is completely symmetric with respect to all the l_i's ($i = 1, 2, \ldots, n$). *) This will be later reconfirmed by working out a closed expression for Z_n (see (2.12)).

The values of Z_2 and Z_3 can be obtained by elementary considerations.

$$\begin{cases} Z_2(l_1, l_2) = \delta_{2l_1, 2l_2} \\ Z_3(l_1, l_2, l_3) = \begin{cases} 0, & \text{if one or three of } l\text{'s are half-odd integers,} \\ \Delta(l_1, l_2, l_2), & \text{otherwise,} \end{cases} \end{cases} \tag{2.3}$$

where

$$\Delta(l_1, l_2, l_3) = \begin{cases} 1, & \text{for } |l_1 - l_2| \leqq l_3 \leqq l_1 + l_2, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

In view of (2.2) and (2.3) it is reasonable to define Z_1 by

$$Z_1(l) \equiv Z_2(l, 0) = \delta_{2l, 0}. \tag{2.6}$$

In a more general case we can evaluate $Z_n(l_1, \ldots, l_n)$ with the help of a well-known method of resolving the product of the characters of the representations

*) This is the reason, why we have employed a symmetrical notation for Z.

$\mathcal{D}^{(l_1)}, \ldots, \mathcal{D}^{(l_{n-2})}$ and $\mathcal{D}^{(l_{n-1})}$, into a sum of characters of irreducible representations (cf. e.g. [2]): The character of $\mathcal{D}^{(l_i)}$ is given by

$$\chi^{(l_i)}(\varphi) = \sum_{\eta=-l_i}^{l_i} \exp(i\eta\varphi) = \frac{\sin(l_i+\frac{1}{2})\varphi}{\sin\frac{\varphi}{2}}. \tag{2.7}$$

We introduce $F^{(n-1)}_{l_1,\ldots,l_{n-1}}(\eta)$ by the equation,

$$\chi^{(l_1)}\cdot\chi^{(l_2)},\ldots,\chi^{(l_{n-1})} = \prod_{i=1}^{n-1}\Big|\sum_{\nu=-l_i}^{l_i}\exp(i\nu\varphi)\Big| = \sum_{2\eta=-\infty}^{\infty} F^{(n-1)}_{l_1,\ldots,l_{n-1}}(\eta)\cdot\exp(i\eta\varphi). \tag{2.8}$$

Evidently,

$$F^{(n-1)}_{l_1,\ldots,l_{n-1}}(-\eta) = F^{(n-1)}_{l_1,\ldots,l_{n-1}}(\eta), \tag{2.9}$$

and

$$F^{(n-1)}_{l_1,\ldots,l_{n-1}}(\eta) = 0, \quad \text{for} \quad |\eta| > \sum_{i=1}^{n-1} l_i. \tag{2.10}$$

Then our $Z_n(l_1,\ldots,l_{n-1},l_n)$ is expressed in terms of $F^{(n-1)}_{l_1,\ldots,l_{n-1}}(\eta)$ by

$$Z_n(l_1,\ldots,l_{n-1},l_n) = F^{(n-1)}_{l_1,\ldots,l_{n-1}}(l_n) - F^{(n-1)}_{l_1,\ldots,l_{n-1}}(l_n+1). \tag{2.11}$$

From (2.7), (2.8) and (2.11) we can work out a more closed expression for $Z_n(l_1,\ldots,l_{n-1},l_n)$:

$$Z_n(l_1,\ldots,l_{n-1},l_n) = \frac{1}{2\pi}\int_{-2\pi}^{\pi} d\varphi \frac{\prod_{i=1}^{n}\sin(l_i+\frac{1}{2})\varphi}{\left(\sin\frac{\varphi}{2}\right)^{n-2}} \qquad \frac{2}{\pi}\int_{0}^{\pi} d\varphi \frac{\prod_{i=1}^{n}\sin(2l_i+1)\varphi}{(\sin\varphi)^{n-2}}. \tag{2.12}$$

In this form the complete symmetry of $Z_n(l_1,\ldots,l_{n-1},l_n)$ with respect to all its arguments is obvious. (2.3), (2.4), and (2.6) are included in (2.12) as special cases.

Instead of the usual form (2.11), we can use either of the following modified expressions:

$$Z_n(l_1,\ldots,l_{n-1},l_n) = F^{(n)}_{l_1,\ldots,l_{n-1},l_n}(0) - F^{(n)}_{l_1,\ldots,l_{n-1},l_n}(1), \tag{2.13}$$

or *)

$$Z_n(l_1,\ldots,l_{n-2},l_{n-1},l_n) = F^{(n-2)}_{l_1,\ldots,l_{n-2}}(l_n-l_{n-1}) - F^{(n-2)}_{l_1,\ldots,l_{n-2}}(l_n+l_{n-1}+1). \tag{2.14}$$

(2.13) is obvious from (2.2) and (2.11), while (2.14) can be derived with the help of the relation

$$F^{(n-1)}_{l_1,\ldots,l_{n-1}}(\eta) = \sum_{\nu} F^{(n-2)}_{l_1,\ldots,l_{n-2}}(\eta-\nu)\,F^{(1)}_{l_{n-1}}(\nu) \qquad \sum_{\nu=-l_{n-1}}^{l_{n-1}} F^{(n-2)}_{l_1,\ldots,l_{n-2}}(\eta-\nu). \tag{2.15}$$

*) The formula (2.14) is practically useful, when we evaluate Z_n's in a straightforward way. See appendix to the second note.

If we put (2.15), namely, into (2.11), all the terms cancel each other, except two, which stand in (2.14).

These modified formulae are of course completely equivalent to (2.11) and lead again to (2.12). When we are interested in an approximate evaluation of Z's, however, some of them turn out more convenient than the others, depending on the values of l_i's (see the second note).

3. Recurrence formula

The $Z_n(l_1, l_2, ..., l_{n-1}, l_n)$'s obey the following recurrence formula:

$$(3.1)\quad Z_n(l_1, l_2, ..., l_j, l_{j+1}, ..., l_{n-1}, l_n) = \sum_{2\lambda=0}^{\infty} Z_{j+1}(l_1, l_2, ..., l_j, \lambda) \cdot Z_{n-j-1}(\lambda, l_{j+1}, ..., l_n).$$

$$\text{for} \quad j = 1, 2, ..., n-1.$$

Proof. With the help of (2.12),

$$Z_{j+1}(l_1, l_2, ..., l_j, \lambda) = \frac{1}{\pi} \int_{-\pi}^{\pi} d\varphi \, \frac{\prod_{i=1}^{j} \sin(2l_i + 1)\varphi}{(\sin\varphi)^{j-1}} \sin(2\lambda + 1)\varphi,$$

$$Z_{n-j-1}(\lambda, l_{j+1}, ..., l_n) = \frac{1}{\pi} \int_{-\pi}^{\pi} d\xi \, \frac{\prod_{i=j+1}^{n} \sin(2l_i + 1)\xi}{(\sin\xi)^{n-j-1}} \sin(2\lambda + 1)\xi.$$

Putting these expressions into the left-hand side of (5.1) and applying the relation

$$\sum_{2\lambda=0}^{\infty} \sin(2\lambda+1)\varphi \cdot \sin(2\lambda+1)\xi = \frac{\pi}{2} \{\delta(\varphi - \xi) - \delta(\varphi + \xi)\},$$

we obtain the right-hand side.

By repeated application of this recurrence formula we find,

$$(3.2)\quad Z_n(l_1, ..., l_n) = \sum_{2\lambda_1} \cdots \sum_{2\lambda_{n-3}} Z_3(l_1, l_2, \lambda_1) \cdot Z_3(\lambda_1, l_3, \lambda_2) \times$$

$$\times Z_3(\lambda_2, l_4, \lambda_3) \ldots Z_3(\lambda_{n-4}, l_{n-2}, \lambda_{n-3}) \cdot Z_3(\lambda_{n-3}, l_{n-1}, l_n).$$

Remembering the property of Z_3, (2.4), we see that, in the case of integer l_i's, the above summand is equal to one when

$$|l_2 - \lambda_1| \leqq l_1 \leqq l_2 + \lambda_1,$$

.....

$$|l_{j+2} - \lambda_{j+1}| \leqq \lambda_j \leqq l_{j+2} + \lambda_{j+1}, \quad (j = 1, 2, ..., n-4)$$

.....

$$|l_{n-1} - l_n| \leqq \lambda_{n-3} \leqq l_{n-1} + l_n,$$

and vanishes otherwise. This justifies that such a summation has been replaced by Z in deriving the formula (1.1) in [1]. *) The same procedure leads to (1.2), with an additional condition that, if there is an odd number of half-odd integer arguments in Z, it must vanish.

Another example of the application of the recurrence formula is the separation of the total spin angular momentum:

$$(3.3)\qquad Z_{2N+1}(l_1, \sigma_1, l_2, \sigma_2, ..., l_N, \sigma_N, J) = \sum_{L,S} Z_3(L, S, J) \cdot Z_{N+1}(\sigma_1, ..., \sigma_N, S) \cdot Z_{N+1}(l_1, ..., l_N, L),$$

where more conventional notation for the total angular momentum (J), the total orbital argular momentum (L), the total spin (S) has been used.

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Summary

This note describes general properties, methods of evaluation, and a recurrence formula of the angular momentum weight factor $Z_n(l_1, l_2, ..., l_n)$, which plays an essential role in the statistical treatment of the multiple particle production with angular momentum conservation, and which is defined by:

$$\mathcal{D}^{(l_1)} \times \mathcal{D}^{(l_2)} \times ... \times \mathcal{D}^{(l_{n-1})} = \sum_{\lambda} Z_n(l_1, l_2, ..., l_{n-1}, \lambda) \cdot \mathcal{D}^{(\lambda)}.$$

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REFERENCES

[1] Z. Koba, Acta Physica Polonica, **20** [to be published]. Report No. 178/VII, Institute of Nuclear Research, Warsaw, October, 1960. [A short summary in:] Nuovo Cim., **3** (1960), 608.

[2] P. Wigner, *Group theory and its application to the quantum mechanics of atomic spectra*, New York and London, 1959.

*) See (2.11) and (2.18) of [1]. We get the expressions in [1] by performing the following substitution: $n \to N+1$, $l_1 \to L$, $l_{j+1} \to l_j$, $(j = 1, 2, ..., N)$.

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Milne's Problem For Two Adjacent Half Spaces

by

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Presented by L. INFELD on January 25, 1961

Introduction

The purpose of this paper is to apply the method developed by K. M. Case [1] in the one-velocity neutron transport theory to the Milne's problem for two adjacent half spaces, characterized by different properties with respect to the migration of neutrons.

Formulation of the problem

There are two half spaces characterized by quantities c, l and c^*, l^* for $x > 0$ and $x < 0$, respectively. Let us assume that in each half space the isotropic and uniform sources S and S^* are distributed (the generalization for the case of arbitrary sources is by the help of Green function given by K. M. Case [1] quite straightforward). At both infinities act the appropriate plane sources. Our aim is to solve the one-velocity Boltzmann equation in plane geometry in two considered regions:

$$\mu \frac{\partial \psi}{\partial x} + \psi = \frac{c}{2} \int_{-1}^{1} \psi(x, \mu')\, d\mu' + S \qquad \text{for } x > 0 \tag{1}$$

and

$$\mu \frac{\partial \psi^*}{\partial x} + \psi^* = \frac{c^*}{2} \int_{-1}^{1} \psi^*(x, \mu')\, d\mu' + S^* \qquad \text{for } x < 0, \tag{2}$$

where x is measured in l and l^* units, respectively.

These two solutions must satisfy the following boundary conditions:

a) $\psi(x, \mu) \underset{x \to +\infty}{\longrightarrow} e^{\frac{x}{\nu_0}} \varphi_-(\mu)$ (the plane source at + infinity)

b) $\psi^*(x, \mu) \underset{x \to -\infty}{\longrightarrow} e^{-\frac{x}{\nu_0^*}} \varphi_+^*(\mu)$ (the plane source at — infinity)

c) $\psi(0, \mu) = \psi^*(0, \mu)$ (the continuity at the dividing surface).

The general form of the solution of the one-velocity Boltzmann equation with isotropic and uniform source is [1]:

$$\psi(x,\mu) = a\,\varphi_+(\mu)\,e^{-\frac{x}{\nu_0}} + b\,\varphi_-(\mu)\,e^{\frac{x}{\nu_0}} + \int_{-1}^{1} A(\nu)\,\varphi_\nu(\mu)\,d\nu + \psi_s, \tag{3}$$

where ν_0 is the positive root of the equation

$$1 - c\,\nu\,\operatorname{arth}\frac{1}{\nu} = 0 \tag{4}$$

and

$$\varphi_\pm(\mu) = \frac{c}{2}\frac{\nu_0}{\nu_0 \pm \mu}; \qquad \varphi_\nu(\mu) = \frac{c}{2}\frac{P}{\nu - \mu} + \lambda(\nu)\,\delta(\nu - \mu), \tag{5}$$

$$\lambda(\nu) = 1 - c\nu\,\operatorname{arth}\nu, \tag{6}$$

$$\psi_s = \frac{S}{1-c}. \tag{7}$$

Completely analogical expression should be written for $\psi^*(x,\mu)$.

Using conditions a) and b) of our set of boundary conditions we can write the condition c) in the form:

$$\varphi_+^*(\mu) + b^*\varphi_-^*(\mu) + \int_{-1}^{0} A^*(\nu)\,\varphi_\nu^*(\mu)\,d\nu + \psi_s^* = a\,\varphi_+(\mu) + \tag{8}$$

$$+ \varphi_-(\mu) + \int_0^1 A(\nu)\,\varphi_\nu(\mu)\,d\nu + \psi_s.$$

We rewrite this equation in the form

$$\int_{-1}^{1} \chi(-\nu)\,A^*(\nu)\,\varphi_\nu^*(\mu)\,d\nu - \int_{-1}^{1} \chi(\nu)\,A(\nu)\,\varphi_\nu(\mu)\,d\nu = h(\mu) - b^*\varphi_-^*(\mu) - a\,\varphi_+(\mu), \tag{9}$$

where $$h(\mu) = \psi_s - \psi_s^* + \varphi_-(\mu) - \varphi_+^*(\mu)$$

and

$$\chi(\nu) = \begin{cases} 1 & \nu > 0 \\ 0 & \nu < 0. \end{cases}$$

Solution of the problem

Let us multiply Eq. (9) by $\mu\,\varphi_\nu^*(\mu)$ and integrate with respect to μ in the interval $(-1, 1)$. Using the following properties of functions $\varphi_\pm(\mu)$, $\varphi_\pm^*(\mu)$, $\varphi_\nu(\mu)$, $\varphi_\nu^*(\mu)$ [2]:

$$\int_{-1}^{1} \mu\,\varphi_\pm(\mu)\,\varphi_\nu^*(\mu)\,d\mu = \frac{c - c^*}{c}\,\nu\,\varphi_\pm(\nu), \tag{10}$$

$$\int_{-1}^{1} \mu\,\varphi_\nu(\mu)\,d\mu \int_{-1}^{1} Q(\eta)\,\varphi_\eta^*(\mu)\,d\eta = \nu\left[\lambda(\nu)\,\lambda^*(\nu) + \frac{\pi^2 c c^* \nu^2}{4}\right] Q(\nu) + \tag{11}$$

$$+ \frac{c^* - c}{2}\int_{-1}^{1} \frac{\eta\,Q(\eta)}{\eta - \nu}\,d\eta,$$

where $Q(\eta)$ is an arbitrary function, we obtain the following equation:

$$(12)\quad \mu\left[\lambda^{*2}(\mu)+\frac{\pi c^{*2}\mu^2}{4}\right]\chi(-\mu)A(\mu)-\mu\left[\lambda(\mu)\lambda^*(\mu)-\frac{\pi^2 cc^*\mu^2}{4}\right]\chi(\mu)A(\mu)$$
$$-\frac{c-c^*}{2}\mu\int_{-1}^{1}\frac{\nu\chi(\nu)A(\nu)}{\nu-\mu}d\nu=\int_{-1}^{1}\eta\varphi_\eta^*(\eta)h(\eta)d\eta+a\frac{c-c^*}{c}\mu\varphi_+(\mu).$$

By specialization of μ and by dividing both sides by μ we obtain:

$$(13)\quad \left[\lambda(\mu)\lambda^*(\mu)+\frac{\pi^2 cc^*\mu^2}{4}\right]A(\mu)+\frac{c-c^*}{2}\int_{-1}^{1}\frac{\nu A(\nu)}{\nu-\mu}d\nu$$
$$=-\frac{1}{\mu}\int_{-1}^{1}\eta h(\eta)\varphi_\eta^*(\eta)d\eta-a\frac{c-c^*}{c}\varphi_+(\mu);\qquad \mu>0$$

$$(14)\quad \left[\lambda^{*2}(\mu)+\frac{\pi^2 c^*\mu^2}{4}\right]A^*(-\mu)+\frac{c-c^*}{2}\int_{-1}^{1}\frac{\nu A(\nu)}{\nu+\mu}d\nu=$$
$$=-\frac{1}{\mu}\int_{-1}^{1}\eta h(\eta)\varphi_{-\eta}^*(\eta)d\eta+a\frac{c-c^*}{c}\varphi_+(-\mu);\qquad \mu>0$$

Eq. (13), which is singular integral equation in the dominant form, is solved in [2]. The result is:

$$(15)\quad A(\mu)=\frac{Q(\mu)}{Q^2(\mu)-P^2(\mu)}f(\mu)+$$
$$+\frac{1}{\pi i[Q(\mu)-P(\mu)]x_0^-(\mu)}\int_0^1\frac{x_0^-(\nu)P(\nu)f(\nu)d\nu}{[Q(\nu)+P(\nu)](\nu-\mu)},$$

where

$$(16)\quad x_0^-(\mu)=\left\{\frac{Q(\mu)-P(\mu)}{Q(\mu)+P(\mu)}\right\}^{1/2}\exp\left\{\frac{1}{\pi}P\int_0^1\frac{\Theta(\nu)}{\nu-\mu}d\nu\right\},$$

$$(17)\quad \begin{cases}\Theta(\nu)=\arg[Q(\nu)-P(\nu)],\\ P(\mu)=\dfrac{i\pi\mu}{2}[c-c^*],\\ Q(\mu)=\lambda(\mu)\lambda^*(\mu)+\dfrac{\pi^2 cc^*}{4}\mu^2,\\ f(\mu)=-\dfrac{1}{\mu}\displaystyle\int_{-1}^{1}\eta\varphi_\eta^*(\eta)h(\eta)d\eta-a\frac{c-c^*}{c}\varphi_+(\mu).\end{cases}$$

It is worth noting that this singular integral equation gives no additional condition.

The multiplication of Eq. (9) by $\mu\,\varphi^*_{\pm}(\mu)$ and integration with respect to μ in the interval $(-1, 1)$ gives us the next two relations:

$$(18)\qquad a\frac{c-c^*}{2}\frac{\nu_0\nu_0^*}{\nu_0^*-\nu_0}=-\frac{c^*-c}{c^*}\int_0^1 \nu A(\nu)\varphi^*_-(\nu)\,d\nu-\int_{-1}^{1}\mu h(\mu)\varphi^*_+(\mu)\,d\mu,$$

$$(19)\qquad b^*\frac{c^*}{2}\nu_0^{*3}\left[\frac{\nu_0^{*2}-1}{c^*}-\frac{1}{\nu_0^{*2}}\right]+a\,\frac{c^*-c}{2}\frac{\nu_0\nu_0^*}{\nu_0^*+\nu_0}=-\frac{c^*-c}{c^*}\int_0^1 \nu A(\nu)\varphi^*_-(\nu)\,d\nu$$
$$-\int_{-1}^{1}\mu h(\mu)\varphi^*_-(\mu)\,d\mu,$$

in derivation of which we have used the following formulas [2]:

$$(20)\qquad \begin{cases} \displaystyle\int_{-1}^{1}\mu\varphi^*_{\pm}(\mu)\varphi_{\pm}(\mu)\,d\mu=\pm\frac{c^*-c}{2}\frac{\nu_0^*\nu_0}{\nu_0^*-\nu_0}, \\ \displaystyle\int_{-1}^{1}\mu\varphi^*_{\pm}(\mu)\varphi_{\mp}(\mu)\,d\eta=\pm\frac{c-c^*}{2}\frac{\nu_0^*\nu_0}{\nu_0^*+\nu_0}, \\ \displaystyle\int_{-1}^{1}\mu\varphi^*_{\pm}(\mu)\varphi^*_{\pm}(\mu)\,d\mu=\pm\frac{c^*}{2}\nu_0^{*3}\left[\frac{c^*}{\nu_0^{*2}-1}-\frac{1}{\nu_0^{*2}}\right]. \end{cases}$$

Having solved Eq. (13) we have now to our disposition three equations, namely (14), (18), (19), to determine the remaining three unknowns: $A^*(\mu)$, which is determined by Eq. (14), and a and b^*, determined by the last two equations. In this manner all coefficients in expression (3) and in analogical expression for φ^* are determined and the Milne's problem for two adjacent half spaces is solved completely.

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REFERENCES

[1] K. M. Case, Annals of Physics, **9** (1960), 1.

[2] A. Kuszell, *Report of the Institute for Nuclear Research of the Polish Academy of Sciences* No. 212 (1961) [submitted to Acta Physica Polonica].

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THEORETISCHE PHYSIK

Quantentheoretische Ableitung der magnetischen Struktur anisotroper ferromagnetischer Einkristalle

von

W. ZIĘTEK

Vorgelegt von W. RUBINOWICZ am 25 Januar 1961

1. Bekanntlich entstehen im ferromagnetischen Einkristall, dessen Temperatur wesentlich niedriger als die Curie-Temperatur ist, sogenannte Elementarbezirke, die homogen fast bis zur Sättigung magnetisiert sind, aber verschiedene Magnetisierungsrichtungen aufweisen. Die Kenntniss dieser Richtungen in Bezug auf die Kristallachsen, der Dimensionen und Energien der Elementarbezirke sowie der Grenzgebiete zwischen ihnen, in denen die Änderung der Magnetisierungsrichtung hauptsächlich vor sich geht, ist für die Praxis von grosser Bedeutung. Fast alle bisherigen Lösungen dieses Problems (s.z.B. [1]—[3]) stützen sich bedeutsam auf klassische Annahmen. Die einzige konsequent quantentheoretische Betrachtung hat Ogutschi [4] durchgeführt, jedoch nur an einem eindimensionalen Spinkettenmodell.

Die vorliegende Arbeit gibt ein recht einfaches rein quantentheoretisches Minimalisationsverfahren für die Lösung dieses Problems, das praktisch auf alle typischen Kristallgitter angewendet werden kann und die meisten experimentell beobachteten magnetischen Strukturen in den wichtigsten Kristallrichtungen zu erhalten erlaubt. Dabei berücksichtigen wir die Anisotropie im Hamiltonoperator einheitlich durch Multipolwechselwirkungsglieder, deren phänomenologische Kopplungskonstanten sich durch das Heisenbergsche Austauschintegral ausdrücken lassen [5] und nötigenfalls auch ohne weiteres die Magnetostriktion zu berücksichtigen erlauben, die wir hier vernachlässigen. Auch betrachten wir einfachheitshalber nur die Wechselwirkung unmittelbar benachbarter Gitteratome und setzen das Austauschintegral als konstant voraus. (Für hexagonale Gitter muss man selbstverständlich zwei Austauschintegrale einführen, die Rechnungen sind aber grundsätzlich dieselben).

2. Der Hamiltonoperator habe die Form

$$
(1)\quad \begin{cases} H = B \sum\limits_{a} \mathcal{H}_{\mu} S^{a}_{\mu} + \sum\limits_{<\alpha\beta>} P^{\alpha\beta}_{\mu_1\mu_2} S^{a}_{\mu_1} S^{\beta}_{\mu_2} - \sum\limits_{<\alpha\beta>} Q^{\alpha\beta}_{\mu_1\mu_2\mu_3\mu_4} S^{a}_{\mu_1} S^{a}_{\mu_2} S^{\beta}_{\mu_3} S^{\beta}_{\mu_4}, \\ P^{\alpha\beta}_{\mu_1\mu_2} = (A + C)\,\delta_{\mu_1\mu_2} - 3C\,(\varrho^{\alpha\beta})^{-2}\,\varrho^{\alpha\beta}_{\mu_1}\,\varrho^{\alpha\beta}_{\mu_2}, \\ Q^{\alpha\beta}_{\mu_1\mu_2\mu_3\mu_4} = D\,(\varrho^{\alpha\beta})^{-4}\,\varrho^{\alpha\beta}_{\mu_1}\,\varrho^{\alpha\beta}_{\mu_2}\,\varrho^{\alpha\beta}_{\mu_3}\,\varrho^{\alpha\beta}_{\mu_4}, \end{cases}
$$

wobei folgende Bezeichnungen verwendet werden: α, β — Indizes der Gitteratome, $\sum_{\langle\alpha\beta\rangle}$ — Summe über unmittelbar benachbarte Atome, $\mu = 1, 2, 3$ — Nummerierung der Komponenten (über sich wiederholende Komponentenindizes ist zu summieren), $\mathcal{H}_\mu$ — äusseres Magnetfeld, $\varrho_\mu^{\alpha\beta}$ — Ortsvektor vom Atom α zum unmittelbar benachbarten β, $-A$, $-B$, C, D — positive Konstanten, S_μ^α — Spinoperator mit den üblichen Vertauschungsrelationen

$$S_{\mu_1}^\alpha S_{\mu_2}^\beta - S_{\mu_2}^\beta S_{\mu_1}^\alpha = i\,\delta^{\alpha\beta}\,\varepsilon_{\mu_1\mu_2\mu_3} S_{\mu_3}^\alpha ,$$

wo $\varepsilon_{\mu_1\mu_2\mu_3}$ den antimetrischen Einheitspseudotensor bezeichnet.

Wir führen den Grundzustand $|-S\rangle$ des isotropen ferromagnetischen Einkristalls ein (s.z.B. [6])

$$S_3^\alpha\,|-S\rangle = -S\,|-S\rangle, \qquad S_\mu^\alpha S_\mu^\alpha = S(S+1), \qquad S > \tfrac{1}{2} \tag{2}$$

und definieren folgende unitäre Transformation

$$U = WV = \exp\left\{ i \sum_\alpha \varphi^\alpha d_{\mu_1}^\alpha S_{\mu_1}^\alpha \right\} \cdot \exp\left\{ i \sum_\alpha \vartheta^\alpha e_{\mu_2}^\alpha S_{\mu_2}^\alpha \right\}, \tag{3}$$

die mit $e_\mu^\alpha e_\mu^\alpha = d_\mu^\alpha d_\mu^\alpha = 1$ erstmals Drehungen um die Achsen mit den Richtungscosinusen e_μ^α um die Winkel ϑ^α bewirkt, und dann Drehungen um die Achsen mit den Richtungscosinusen d_μ^α um die Winkel φ^α.

Jetzt machen wir als erste Näherung die Annahme, dass *der Zustand* $|\varphi^\alpha, \vartheta^\alpha\rangle \equiv U\,|-S\rangle$ *mit solchen* φ_0^α, ϑ_0^α, *für die*

$$\langle \varphi_0^\alpha, \vartheta_0^\alpha | H | \varphi_0^\alpha, \vartheta_0^\alpha \rangle = \min_{\varphi^\alpha,\,\vartheta^\alpha} \langle \varphi^\alpha, \vartheta^\alpha | H | \varphi^\alpha, \vartheta^\alpha \rangle , \tag{4}$$

*gut den Grundzustand des anisotropen ferromagnetischen Einkristalls annähert**) **).

Bei dieser Annahme stützen wir uns einerseits auf die experimentellen und bisherigen theoretischen Resultate auf diesem Gebiet und andererseits auf die vielfach mit Erfolg verwendete Variationsnäherungsmethode.

3. Wegen $\tilde{S}_\mu^\alpha = U S_\mu^\alpha U^+ = R_{\mu\nu}^\alpha S_\nu^\alpha$ haben wir für (1)

$$\begin{cases} UHU^+ = B \sum_\alpha \tilde{\mathcal{H}}_\nu^\alpha S_\nu^\alpha - \sum_{\langle\alpha\beta\rangle} \tilde{P}_{\nu_1\nu_2}^{\alpha\beta} S_{\nu_1}^\alpha S_{\nu_2}^\beta + \sum_{\langle\alpha\beta\rangle} \tilde{Q}_{\nu_1\nu_2\nu_3\nu_4}^{\alpha\beta} S_{\nu_1}^\alpha S_{\nu_2}^\alpha S_{\nu_3}^\beta S_{\nu_4}^\beta , \\ \tilde{\mathcal{H}}_\nu^\alpha = \mathcal{H}_\mu R_{\mu\nu}^\alpha , \quad \tilde{P}_{\nu_1\nu_2}^{\alpha\beta} = P_{\mu_1\mu_2}^{\alpha\beta} R_{\mu_1\nu_1}^\alpha R_{\mu_2\nu_2}^\beta , \\ \tilde{Q}_{\nu_1\nu_2\nu_3\nu_4}^{\alpha\beta} = Q_{\mu_1\mu_2\mu_3\mu_4}^{\alpha\beta} R_{\mu_1\nu_1}^\alpha R_{\mu_2\nu_2}^\alpha R_{\mu_3\nu_3}^\beta R_{\mu_4\nu_4}^\beta . \end{cases} \tag{5}$$

Wenn wir noch in (5) die Operatoren S_+^α, S_-^α einführen [6], so bekommen wir für (4) den einfachen Ausdruck

---

*) Die Wahl der Richtungscosinuse der Drehachsen hängt von der jeweils betrachteten Kristallrichtung ab. Für die meisten Fälle können sie sogar von α unabhängig sein.

**) Bei weiteren Näherungen kann man ausser dem Zustand $|-S\rangle$ Zustände mit einer, oder mehreren Spinwellen als Ausgangspunkt nehmen (über Spinwellen siehe z.B. [6]).

$$(6)\qquad \begin{cases} h \equiv \langle -S|UHU^{+}|-S\rangle = -BS\sum\limits_{\alpha}\tilde{\mathcal{H}}_3^{\alpha} + S^2\sum\limits_{\langle\alpha\beta\rangle}\{\tilde{P}_{33}^{\alpha\beta}+\tilde{Q}^{\alpha\beta}\}, \\ \tilde{Q}^{\alpha\beta} = \frac{1}{4}\{\tilde{Q}_{1111}^{\alpha\beta}+\tilde{Q}_{1122}^{\alpha\beta}+\tilde{Q}_{2211}^{\alpha\beta}+\tilde{Q}_{2222}^{\alpha\beta}\}+ \\ \qquad\qquad +\frac{1}{2}S\{\tilde{Q}_{1133}^{\alpha\beta}+\tilde{Q}_{3311}^{\alpha\beta}+\tilde{Q}_{2233}^{\alpha\beta}+\tilde{Q}_{3322}^{\alpha\beta}\}+S^2\tilde{Q}_{3333}^{\alpha\beta}. \end{cases}$$

4. Jetzt kann man den Minimalwert von $h(\varphi^{\alpha}, \vartheta^{\alpha})$, also (4), auf zwei Arten bestimmen:

a) fasst man h als Funktion der $2N$ Variablen $\varphi^{\alpha}, \vartheta^{\alpha}$ auf, wo N die Anzahl der Gitteratome ist, so hat man die Gleichungen

$$(7)\qquad \frac{\partial h}{\partial\varphi^{\alpha}} = \frac{\partial h}{\partial\vartheta^{\alpha}} = 0$$

zu lösen;

b) macht man einen entsprechenden Grenzübergang $\varphi^{\alpha}\to\varphi(\dot{r})$ und $\vartheta^{\alpha}\to\vartheta(\dot{r})$, so kann man h als Funktional auf $\varphi(x_{\mu})$ und $\vartheta(x_{\mu})$ ansehen und das übliche Variationsverfahren anwenden: $\delta h = 0$.

Selbstverständlich sind in beiden Fällen die Gleichungen sehr kompliziert und man muss gewisse Näherungen machen und entsprechende Randbedingungen aufstellen, um die Rechnungen zu Ende zu führen. Jedoch haben sich in allen bisher von uns berechneten Fällen beide Verfahren immer als äquivalent erwiesen. Es ist hervorzuheben, dass unsere Methode schon in einfachster Näherung zu allen bisher bekannten Resultaten führt [1]—[3], daher kann man sie auch mit Recht als quantentheoretische Rechtfertigung der bisherigen Rechnungsverfahren betrachten.

5. Wir geben hier beispielsweise die Endresultate für die berechnete magnetische Struktur im einfachen kubischen Kristall mit der Gitterkonstante a in der als X-Achse angenommenen Kristallrichtung [100] an. Dabei wurde vorausgesetzt: $d_2^n = d_2^a = \vartheta^{\alpha} = \mathcal{H}_{\mu} = 0$, $d_1 = 1$, $\varphi^{\alpha}\to\varphi(x)$, der Kristall ist in der YZ-Ebene unendlich gross). Da gibt es zwei mögliche Resultate:

a) für die in [2] angenommenen Grenzbedingungen $\varphi(\pm\infty) = \pm\pi/4$, $\varphi'(\pm\infty) = 0$ — zwei unendlich grosse fast bis zur Sättigung magnetisierte Bereiche, deren Magnetisierungsrichtungen um den Winkel $\pi/2$ verschoben sind, wobei die Richtungsänderung nach der Gleichung

$$(8)\qquad \sin 2\varphi = \operatorname{th} 2x\sqrt{-\varkappa}, \qquad \varkappa = \frac{D\,1(2S-1)^2}{4a^2(A+C)}$$

vor sich geht, (siehe Abb. 1). Das Intervall $\delta = (-\varkappa)^{-1/2}$ lässt sich mit den Schätzungsannahmen [5] $C \sim |A|(g-2)^2$, $D \sim |A|(g-2)^4$ (g — Landéscher Faktor) einfach ausdrücken

$$(9)\qquad \delta \simeq 4a(g-2)^{-2}\,1\,(2S-1)^{-1}\,2^{-1/2},$$

was für $g = 1{,}9$ und $S = 1$ zu $\delta \simeq 600\,a$ führt und für $S = 2$ zu $\delta \simeq 200\,a$;

b) für die in [3] angenommenen periodischen Bedingungen $\varphi(\pm n\Delta) = \pm n\pi/2$ — fast bis zur Sättigung magnetisierte Bereiche von der Breite Δ, deren Magnetisierungs-

richtungen nacheinander um den Winkel $\pi/2$ verschoben sind, wobei die Richtungsänderung nach der Gleichung

$$\sin 2\varphi = \sin \xi, \qquad \xi = x \cdot \text{const} \tag{10}$$

vor sich geht, (siehe Abb. 2):

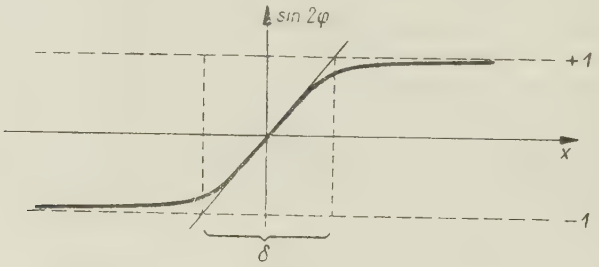

Abb. 1

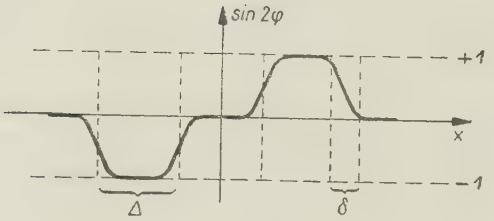

Abb. 2

kann: d.h. die Magnetisierungsrichtungen der Elementarbezirke von der Breite Δ drehen sich in der YZ-Ebene von Bezirk zu Bezirk um den Winkel $\pi/2$; dabei vollzieht sich die Drehung fast ausschliesslich in den Intervallen von der Breite δ (sogenannte Blochwände).

Ein ausführlicher Bericht über die Resultate der Anwendung unserer Methode wird in Acta Physica Polonica erscheinen.

Herrn Professor R. S. Ingarden danke ich für die Anregung zu dieser Arbeit und für manchen Meinungsaustausch während ihrer Anfertigung.

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SCHRIFTTUM

[1] F. Bloch, Zs. f. Phys., **74** (1932), 295.
[2] L. Landau u. E. Lifschitz, Phys. Zs. Sov., **8** (1935), 153; E. Lifschitz, Z.E.T.F., **15** (1945), 97.
[3] M. Schirobokov, DAN, **24** (1939), 426; Ż.E.T.F., **11** (1941), 554; **15** (1945), 57.
[4] T. Ogutschi, Progr. Theor. Phys., **9** (1953), 7.
[5] Van Vleck, *Conference on magnetism and magnetic materials*, Boston, 1956.
[6] F. J. Dyson, Phys. Rev., **102** (1956), 1217.

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THEORETICAL PHYSICS

Second Order Invariant Differential Equation in the Spinor Space

by

J. MOZRZYMAS and J. RZEWUSKI

Presented by L. INFELD on January 31, 1961

The second order linear differential equation in the spinor space, satisfying the condition of invariance with respect to the group $\boldsymbol{c}\cdot\boldsymbol{c}'\cdot\boldsymbol{a}$. has necessarily the form (cf. [2], Eqs. (2.1) and (2.18))

$$\left[\frac{\partial^2}{\partial z_{\alpha;\beta}\,\partial z^{\alpha;\beta}} - \frac{\varkappa^2}{4}\,\epsilon^2_{;\mu}\, z_{\dot\alpha;\dot\beta}\, z^{\dot\alpha;\dot\beta}\right] f(z_{\alpha;\beta},\, z_{\dot\alpha;\dot\beta}) = 0. \tag{1}$$

Here the $z_{\alpha;\beta}$ ($z_{\dot\alpha;\dot\beta} = z^*_{\alpha;\beta}$; $\alpha, \beta = 1, 2$) are four complex spinor variables. The indices before (after) the semicolon transform according to the unimodular group $\boldsymbol{c}(\boldsymbol{c}')$. The group $\boldsymbol{a}$ is the one-parameter phase-group $z'_{\alpha;\beta} = e^{i\varphi}\, z_{\alpha;\beta}$. $\boldsymbol{c}\cdot\boldsymbol{c}'\cdot\boldsymbol{a}$ denotes the direct product of these three commuting groups. $\epsilon_{;\mu}$ is a constant $\boldsymbol{c}$ — scalar, $\boldsymbol{c}'$ — vector of unit length: $|\epsilon^2_{;\mu}| = 1$. $\varkappa^2$ is a positive constant.

We assume a bilinear connection between the variables $x_{\mu;}$ of Minkowski's space and the spinor variables $z_{\alpha;\beta}$, $z_{\dot\alpha;\dot\beta}$:

$$x_{\alpha\dot\beta;} = z_{\alpha;\rho}\, z_{\dot\beta;\dot\sigma}\, \epsilon^{;\rho\dot\sigma}. \tag{2}$$

Here

$$x_{\mu;} \equiv \sigma^{\alpha\dot\beta;}_{\mu;}\, x_{\alpha\dot\beta;}, \quad \epsilon_{;\mu} \equiv \sigma^{;\alpha\dot\beta}_{;\mu}\, \epsilon_{;\alpha\dot\beta}, \tag{3}$$

with $\sigma^{\alpha\dot\beta}_{\mu}$ the conventional Pauli matrices. One easily verifies that

$$x^2_{\mu;} = \frac{1}{4}\,\epsilon^2_{;\mu}\, |z_{\alpha;\beta}\, z^{\alpha;\beta}|^2. \tag{4}$$

Let us rewrite (1) in the new variables

$$\begin{cases} w_1 = z_{1;1} + z_{2;2} & w_3 = -\,i\,(z_{1;2} + z_{2;1}) \\ w_2 = -\,i\,(z_{1;1} - z_{2;2}) & w_4 = z_{1;2} - z_{2;1}. \end{cases} \tag{5}$$

We obtain

$$\sum_{k=1}^{4}\left(\frac{\partial^2}{\partial w_k^2} - \frac{\varkappa^2}{16}\,\epsilon^2_{;\mu}\, w_k^{*2}\right) f(w_l, w_l^*) = 0. \tag{6}$$

The next step is the introduction of complex angular variables

$$(7)\qquad \begin{cases} w_1 = u \sin\vartheta_1 \sin\vartheta_2 \cos\vartheta_3 \\ w_2 = u \sin\vartheta_1 \sin\vartheta_2 \sin\vartheta_3 \\ w_3 = u \sin\vartheta_1 \cos\vartheta_2 \\ w_4 = u \cos\vartheta_1 \\ u = 2\sqrt{\varrho}\, e^{-i\frac{\varphi}{2}} \\ \vartheta_j = \xi_j + i\eta_j \quad (j = 1, 2, 3). \end{cases}$$

$$(8)\qquad \varrho^2 = \frac{1}{16} u^2 u^{*2} = \frac{1}{4} |z_{\alpha;\beta} z^{\alpha;\beta}|^2 = |x_\mu^2|; \qquad \varrho = \sqrt{|x_\mu^2|}\,.$$

In terms of the angular variables (7) Eq. (6) takes the form

$$(9)\qquad (R + J - \varkappa^2 \epsilon_{;\mu}^2 \varrho^2)\, f(\varrho, \alpha, \vartheta_j, \vartheta_j^*) = 0,$$

where

$$(10)\qquad R = \varrho^2 \frac{\partial^2}{\partial\varrho^2} + 3\varrho \frac{\partial}{\partial\varrho} + 2i\left(\varrho \frac{\partial}{\partial\varrho} + 1\right)\frac{\partial}{\partial\alpha} - \frac{\partial^2}{\partial\alpha^2}$$

and

$$(11)\qquad J = \frac{1}{\sin^2\vartheta_1}\left[\frac{\partial}{\partial\vartheta_1}\left(\sin^2\vartheta_1 \frac{\partial}{\partial\vartheta_1}\right) + \frac{1}{\sin\vartheta_2}\frac{\partial}{\partial\vartheta_2}\left(\sin\vartheta_2 \frac{\partial}{\partial\vartheta_2}\right) + \frac{1}{\sin^2\vartheta_2}\frac{\partial^2}{\partial\vartheta_3^2}\right].$$

Putting

$$(12)\qquad \vartheta_1 = i\eta_1, \qquad \vartheta_2 = \xi_2, \qquad \vartheta_3 = \xi_3,$$

in (11) for points of the inside of the light cone ($\epsilon_{;\mu}^2 = -1$) and

$$(13)\qquad \vartheta_1 = \frac{\pi}{2} + i\eta_1, \qquad \vartheta_2 = \xi_2, \qquad \vartheta_3 = \xi_3,$$

for the points of the outside of the light cone ($\epsilon_{;\mu}^2 = 1$) we obtain two real operators J_1 and J_2:

$$(14)\qquad J_1 = \frac{-1}{\operatorname{sh}^2\eta_1}\left[\frac{\partial}{\partial\eta_1}\left(\operatorname{sh}^2\eta_1 \frac{\partial}{\partial\eta_1}\right) + \frac{1}{\sin\xi_2}\frac{\partial}{\partial\xi_2}\left(\sin\xi_2 \frac{\partial}{\partial\xi_2}\right) + \frac{1}{\sin^2\xi_2}\frac{\partial^2}{\partial\xi_3^2}\right],$$

$$(15)\qquad J_2 = \frac{1}{\operatorname{ch}^2\eta_1}\left[\frac{\partial}{\partial\eta_1}\left(\operatorname{ch}^2\eta_1 \frac{\partial}{\partial\eta_1}\right) - \frac{1}{\sin\xi_2}\frac{\partial}{\partial\xi_2}\left(\sin\xi_2 \frac{\partial}{\partial\xi_2}\right) - \frac{1}{\sin^2\xi_2}\frac{\partial^2}{\partial\xi_3^2}\right].$$

Introducing now, in the case of J_1, the variables

$$(16)\qquad \begin{cases} y_{1;} = r_1 \operatorname{sh}\eta_1 \sin\xi_2 \cos\xi_3 \\ y_{2;} = r_1 \operatorname{sh}\eta_1 \sin\xi_2 \sin\xi_3 \\ y_{3;} = r_1 \operatorname{sh}\eta_1 \cos\xi_2 \\ y_{0;} = \epsilon_{;0}\, r_1 \operatorname{ch}\eta_1 \\ r_{1;} = \varrho = \sqrt{-x_\mu^2}; \end{cases}$$

and, in the case of J_2, the variables

$$(17)\qquad \begin{cases} y_{1;} = r_2 \operatorname{ch} \eta_1 \sin \xi_2 \cos \xi_3 \\ y_{2;} = r_2 \operatorname{ch} \eta_1 \sin \xi_2 \sin \xi_3 \\ y_{3;} = r_2 \operatorname{ch} \eta_1 \cos \xi_2 \\ y_{0;} = \epsilon_{;3}\, r_2 \operatorname{sh} \eta_1 \\ r_2 = \varrho = \sqrt{x_{\mu;}^2} \end{cases}$$

$$(0 \leqslant r_1, r_2 < \infty, \quad 0 \leqslant \eta_1 < \infty, \quad 0 < \xi_2 \leqslant \pi, \quad -\pi < \xi_3 \leqslant \pi),$$

we get in both cases instead of (9) the equation

$$(18)\qquad \left[\left(\frac{\partial}{\partial y_{\mu;}} - i\,\frac{y_{\mu;}}{y_{\sigma;}^2}\,\frac{\partial}{\partial a}\right)^2 - \varkappa^2\right] f(y_{\nu;} a) = 0.$$

The connexion between the variables $y_{\mu;}$ and $x_{\mu;}$ is given in both cases by relation (2). For the inside of the light cone and in the frame of reference in which $\epsilon_{;0} = 1$, $\epsilon_{;i} = 0$, $(i = 1, 2, 3)$ these relations have the form

$$(19)\qquad \begin{cases} \eta_1 = \dfrac{1}{2} \operatorname{Arth} \dfrac{1}{x_{0;}} \sqrt{x_{1;}^2 + x_{2;}^2 + x_{3;}^2} \\ \xi_2 = \dfrac{1}{2}(1 - \epsilon_{;0})\,\pi + \epsilon_{;0} \operatorname{arc\,tg} \dfrac{1}{x_{3;}} \sqrt{x_{1;}^2 + x_{2;}^2} \\ \xi_3 = -\operatorname{arc\,tg} \dfrac{x_{2;}}{x_{1;}} - \epsilon_{;0}\,\dfrac{\pi}{2}\,. \end{cases}$$

For the outside of the light cone and in the particular frame of reference in which $\epsilon_{;3} = \pm 1$, $\epsilon_{;1} = \epsilon_{;2} = \epsilon_{;0} = 0$, they have the form

$$(20)\qquad \begin{cases} \eta_1 = \operatorname{Arsh} \dfrac{x_{0;}}{\sqrt{2(\epsilon_{;3}\, x_{3;} \sqrt{x_{\sigma;}^2 + x_{\sigma;}^2})}}\,, \\ \xi_2 = \operatorname{arc\,tg} \dfrac{\sqrt{x_{1;}^2 + x_{2;}^2}}{\epsilon_{;3}\, x_{3;} + \sqrt{x_{\sigma;}^2}}\,, \\ \xi_3 = \operatorname{arc\,tg} \dfrac{x_{1;}}{x_{2;}}\,. \end{cases}$$

It is seen, that Eq. (9) is an analytical continuation of Eq. (18). Therefore, from any solution of (18) we may obtain a solution of the general equation (9) by analytical continuation. To obtain a solution for the inside of the light cone we must continue analytically on the way leading through (16), (14) and (12). To obtain a solution for the outside the light cone we have to continue on the way leading through (17), (15) and (13). Thus, the problem of the general equation (9) is reduced to the simpler problem of Eq. (18).

The solutions of (18) and their possible physical interpretation were studied in some detail in [1]. The explicit form of the general solutions was obtained by one of us [3] also by solving directly Eq. (1). In [3] one may also find the derivation of relations (19—20).

Summary

It is shown that the general linear second order differential equation in the spinor space, invariant with respect to the group $c \cdot c' \cdot a$ of this space, is an analytical continuation of a special differential equation treated extensively in [1].

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REFERENCES

[1] J. Rzewuski, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **8**, (1960), 777, 783.
[2] — , Acta Phys. Pol. **18** (1959), 549.
[3] J. Mozrzymas, Zesz. Nauk. Uniw. Wroc. [to be published].

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EXPERIMENTAL PHYSICS

Effect of γ ^{60}Co Radiation on the Dielectric Properties of Triglycine Sulphate

by
B. HILCZER

Presented by M. MIĘSOWICZ on December 10, 1960

The changes in the ferroelectric properties of triglycine sulphate resulting from irradiation with γ ^{60}Co were investigated. The temperature dependence of the dielectric permittivity was measured in non-irradiated samples and in those that had received a varying dose of γ-radiation. Moreover, the changes thus produced in the shape of the dielectric hysteresis loop of triglycine sulphate samples were investigated.

The samples were cut from single crystals grown in this laboratory by Mrs J. Stankowska [1]. These were from 1 mm. to 1.6 mm. thick, with contact areas from 7 mm.^{2} to 40 mm.^{2} each, cut perpendiculary to the ferroelectric axis, and coated with colloidal graphite (aquadag) electrodes; the measurements were carried out with the sample in a holder condenser.

The capacity of the samples, of the order of 10 pF at room temperature, was measured with an accuracy of ± 0.1 pF by the resonance method, at a generator frequency of 1 MHz. The samples were placed in a Höppler thermostat that stabilized the temperature within ± 0.5°C. The temperature dependence of the dielectric permittivity was currently determined for a sample irradiated and a control sample, both cut from the same growth pyramid of the crystal and having the same "dielectric past" (aged at the same temperature).

The hysteresis and current loops were investigated with a 50 Hz AC-field at room temperature.

The triglycine sulphate samples were irradiated at room temperature with γ-rays from a ^{60}Co source of 22.7 mC activity in a plane symmetric radiator, and of 210 mC in one of spherical symmetry. The samples irradiated were situated at a point where the momentary dose amounted to 10^3 r/h and 0.4×10^4 r/h, respectively. The time of irradiation of a sample was 2740 hours, corresponding to a dose of the order of 10^6 r.

The temperature dependence of the dielectric permittivity as obtained in the irradiated and control samples of triglycine sulphate samples throughout the tem-

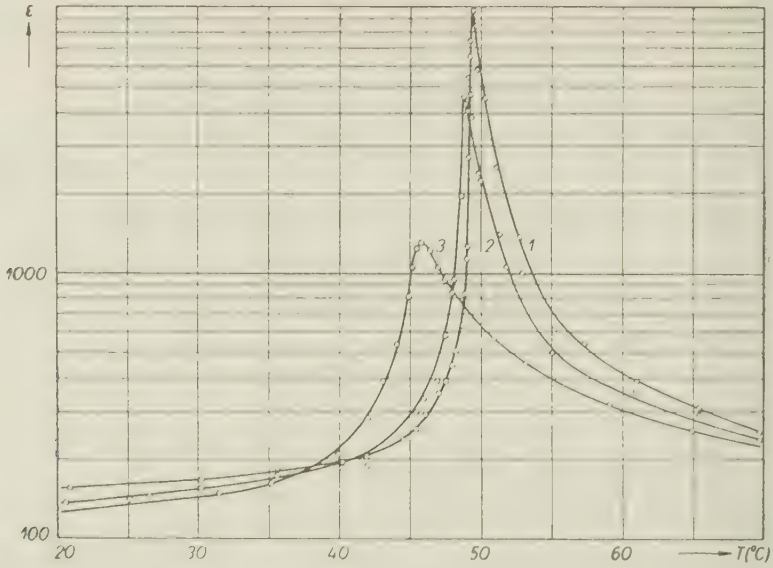

Fig. 1a. Temperature dependence of dielectric permittivity in triglycine sulphate: 1 — non-irradiated (control) sample, 2 — sample irradiated for 344.5 hours in plane radiator, 3 — sample irradiated for 2739 hours (thickness of sample 1.5 mm., area 12.4 mm.²)

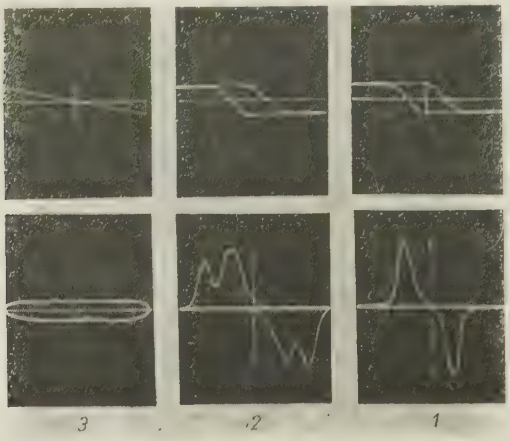

Fig. 1b. Hysteresis loops and current loops corresponding to curves 1—3 in Fig. 1a: $E = 1333$ V/cm.

perature range of 20 – –70 C is shown in Fig. 1a. Curve 1 corresponds to a non-irradiated (control sample, and curves 2 and 3 to samples irradiated in the plane

radiator for 344.5 and 2739 hours, with doses of 0.3×10^6 r and 2.7×10^6 r, respectively. The effect of γ-radiation consists in a decrease in the dielectric permittivity at the Curie point and in a shift of the Curie point towards lower temperatures.

Fig. 1b gives the hysteresis and current loops corresponding to curves 1, 2 and 3 in Fig. 1a. As a result of γ-irradiation, the normal hysteresis loop (1) of the non-irradiated sample assumes the constricted form (2). After sufficiently long irradiation, the triglycine sulphate sample exhibits a linear dependence of the dielectric polarization on the field strength (3). The polarization versus field dependence remains unchanged throughout the temperature range of 20 C—70 C, and its slope is lesser than that obtained with the control sample above the Curie point, corresponding to a lower value of the dielectric permittivity.

Moreover, the temperature hysteresis of the permittivity was measured in non-irradiated and irradiated samples. In either case, when the sample is cooled down slowly from 70°C, the permittivity versus temperature curve lies higher than during heating; moreover, the Curie point is shifted towards lower temperatures.

Fig. 2 shows the shift of the Curie point towards lower temperatures and the relative decrease in the permittivity at the Curie point as depending on the time of irradiation. From Fig. 2a, the shift is seen to be a linear function of the irradiation time. Moreover, in samples irradiated, the Curie-Weiss temperature Θ was found to shift towards lower values. The relative changes in the permittivity at the Curie point due to irradiation (Fig. 2b) are at first large, then diminish, tending to saturation.

Fig. 3 shows oscillograms of the hysteresis loop for a triglycine sulphate crystal, as obtained subsequent to different periods of γ-irradiation in the spherical radiator. As irradiation proceeds, there is at first an increase in the degree of the loop, and later constriction appears. As a result of γ-irradiation, the current loop exhibits la characteristic splitting of the maximum, corresponding to the changes in the hysteresis loop.

From the present investigation, the following statements can be made concerning triglycine sulphate subjected to γ-irradiation from a ^{60}Co source:

1° the dielectric permittivity undergoes a decrease, throughout the temperature range of 20° to 70°C;

2° the relative decrease in permittivity at the Curie point is a non-linear function of the irradiation time;

3° the difference between the Curie point temperature of a non-irradiated and that of an irradiated sample is a linear function of irradiation time;

4 the temperature Θ as determined from the Curie-Weiss law shifts towards lower values in samples previously subjected to irradiation;

5 temperature hysteresis is of the same shape in non-irradiated and irradiated samples;

6° the hysteresis and current loops undergo modification in shape.

The effects of ^{60}Co γ-radiation on the ferroelectric properties of triglycine sulphate are reminiscent of those of UV radiation, X-rays and electrons. The changes in the dielectric properties of triglycine sulphate due to UV irradiation were investigated by Silvestrova and Romaniuk [2], who found a decrease in the dielectric

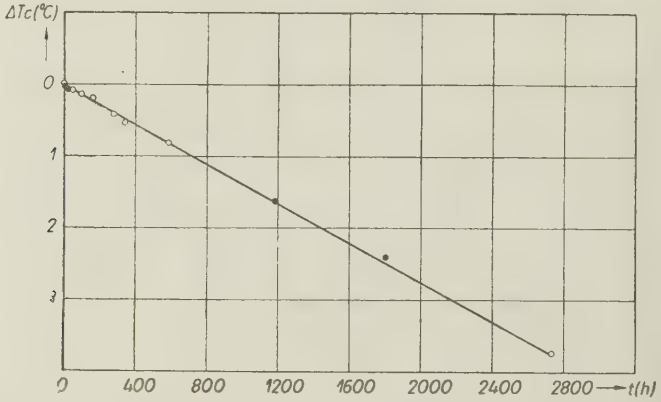

Fig. 2a. Shift in Curie point temperature versus time of irradiation

permittivity between 30° and 70°C, and small variations in the shape of the hysteresis loop. Constriction of the latter due to X-ray and electron (up to 1 MeV) irradiation is also reported by Chynoweth [3].

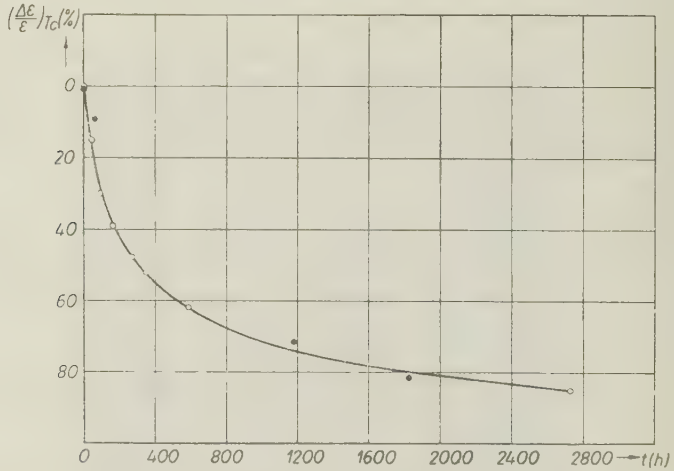

Fig. 2b. Relative variation of the permittivity at Curie point (in %) versus the time of radiation

The effects of ^{60}Co γ-irradiation and X-rays on the ferroelectric properties of Rochelle salt were investigated by Eisner, Zheludyev and Yurin [4]—[6] and by the present author [7], who found a decrease in the permittivity throughout the temperature range of $-25°$ to $+30°C$, a shift of the Curie point towards lower

temperatures, and variations in the shape of the hysteresis loop. These changes resemble those observed in triglycine sulphate as a result of irradiation with a γ ^{60}Co source.

The following explanation is proposed for the effects observed: In the natural state, triglycine sulphate and Rochelle salt present an aproximately antiparallel system of domains, minimum free energy occurring at antiparallel polarization of neighbouring domains. UV, X-ray, γ or electron irradiation of these crystals gives rise to defects therein. The defects introduced during the process of irradiation accumulate in the neighbourhood of the inter-domain walls, diminishing the mobility of the latter. This leads to a decrease in the number of domains participating in the process of switching — the hysteresis loop at first becomes constricted, as in an aged ferroelectric. Subsequent to a long period of irradiation, the dependence

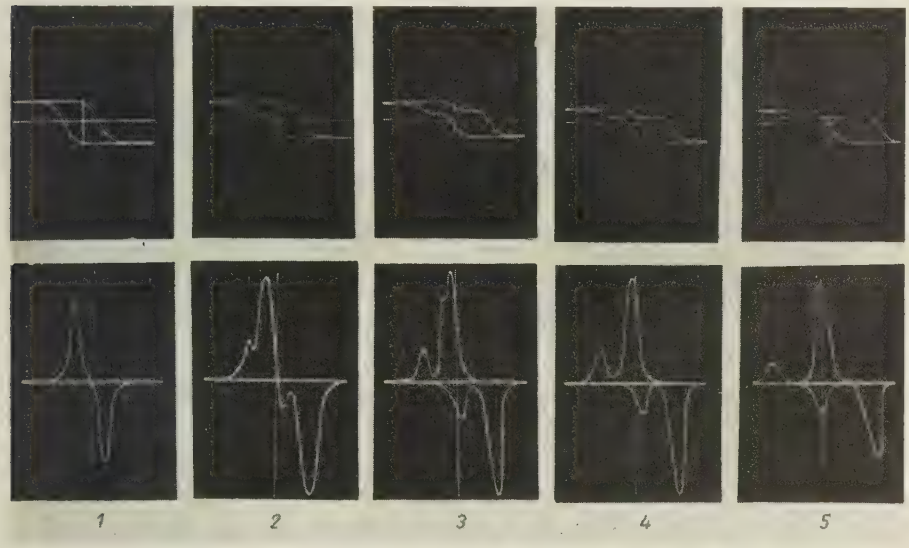

Fig. 3. Hysteresis loops and current loops from triglycine sulphate versus the time of irradiation (spherically symmetrical radiator): 1 — non-irradiated sample, 2 — sample irradiated for 67.5 hours, 3 — 210 hours, r — 283.5 hours, 5 — 448 hours; $E = 2200$ V/cm.

of the polarization on the electric field strength becomes linear. As the number of defects increases, the dielectric permittivity, which depends on the mobility of the inter-domain walls, diminishes. Thus the defects introduced in the process of irradiation can be said to consolidate the state of domain antiferroelectric [8], just as defects introduced into a crystal during growth [9], [10].

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REFERENCES

[1] J. Stankowska, J. Stankowski, Zeszyty Naukowe UAM [in press].
[2] I. M. Silvestrova, N. A. Romaniuk, Kristallografia, **5** (1960), 147.
[3] A. G. Chynoweth, Phys. Rev., **113** (1959), 159.
[4] I. J. Eisner, Izv. Akad. Nauk SSSR, **20** (1956), 215.
[5] I. S. Zheludyev, W. A. Yurin, Izv. Akad. Nauk SSR, **20** (1956), 211.
[6] W. A. Yurin, Izv. Akad. Nauk SSSR, **21** (1957), 329.
[7] B. Maćkowiak 1958 [not published].
[8] Z. Pająk, J. Stankowski, Proc. Roy. Soc., **72** (1958), 1144.
[9] W. P. Konstantinova, W. A. Yurin, Kristallografia, **2** (1957), 294.
[10] J. Stankowska, J. Stankowski, Acta Phys. Polon., **19** (1960), 217.

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EXPERIMENTAL PHYSICS

An Attempt to Obtain Electroluminescence of Organophosphors

by

H. ŁOŻYKOWSKI and H. MĘCZYŃSKA

Presented by A. JABŁOŃSKI on December 28, 1960

Recently a paper was published by A. Tumerman and A. Czayanov on electroluminescence of organophosphors [2]. In this connection it seemed expedient to publish also our results on this subject.

An attempt was made to obtain electroluminescence of the following organic dyes: trypaflavine, acridine yellow, acridine orange, fluorescein, brilliant sulphoflavin FF in an a.c. field.

Coloured cellophane [1] was placed in an electroluminescent cell with two electrodes: one of brass, and the other of semiconducting glass.

The cell thus prepared was inserted into a glass tube, evacuated to a 10^{-5} mm.Hg pressure.

A.c. voltage current applied to the cell produced unhomogeneous weak luminescence observed close to the centre of the cell. If, however, the excitation was produced by UV irradiation, without a.c. field, uniform luminescence of the total surface of the sample was observed.

We investigated also the electroluminescence of an identically prepared cell with an uncoloured cellophane, placed in a 10^{-5} mm.Hg vacuum. The a.c. field produced also in this case a luminescence apparently not differing from that of cellophane with adsorbed dye.

Similar luminescence was observed also, if mica alone was placed in the cell.

The shapes of the brightness waves obtained were similar for all samples investigated.

Our observations agree with those of A. Tumerman and A. Czayanov [2], who investigated the electroluminescence of a cell containing cellophane coloured with acridine orange and with paraffin as dielectric, a cell with paraffin only, with mica, with silicon oil, with transformer oil and with uncoloured cellophane in paraffin

The emission spectra of undyed cellophane and those of cellophane with acridine orange adsorbed, obtained by Tumerman and Czayanov, did not display any significant differences, which would allow to prove the existence of the electroluminescence of acridine orange.

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REFERENCES

[1] A. Bernanose, M. Compte, P. Vouaux, J. de Chim. Phys., **50** (1953), 64.

[2] P. A. Tumerman, B. A. Czayanov, *Investigations on experimental and theoretical physics*. in a coll. work in hour of G. S. Landsberg [in Russian], Moscow, 1959, p. 231.

БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК
СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ
И ФИЗИЧЕСКИХ НАУК

ТОМ IX — Резюме статей — ВЫПУСК 3

Ю. МЕДЕР,

ОЦЕНКА СРЕДНИХ (N, p_n) **ОРТОГОНАЛЬНЫХ РЯДОВ**

стр. 123—127

Пусть $ON\{\varphi_n(x)\}$ обозначает ортонормированную систему, определенную в интервале $\langle 0, 1\rangle$.

Мы скажем, что ортогональный ряд

$$\sum_{n=0}^{\infty} a_n \varphi_n(x) \tag{1}$$

суммируется (N, p_n) к s, если $p_n \geqslant 0$, $p_0 > 0$, $P_n = p_0 + p_1 + \dots p_n \to \infty$ и

$$T_n(x) \frac{1}{P_n}(p_0 s_0(x) + p_1 s_1(x) + \dots + p_n s_n(x)) \to s(x),$$

когда $n \to \infty$, причем $s_n(x)$ обозначает n-тые частичные суммы суммы ряда (1).

При этих обозначениях доказывается следующая теорема:

Теорема 1. *Если* $\{a_n\} \in l^2$, *то*

$$T_n(x) = 0(\log\log P_n), \quad \textit{когда} \quad n \to \infty,$$

почти всегда в $\langle 0, 1\rangle$.

Теорема 2. *Если* $\{v(n)\}$ *обозначает возрастающую последовательность положительных чисел, стремящихся к* ∞ *и если выполнено условие*

$$v(n) = 0(\log\log P_n), \quad \textit{когда} \quad n \to \infty,$$

то существует такая последовательность чисел $\{c_n\}$ *и такая система* $ON\{\Phi_n(x)\}$, *что* $\{c_n\} \in l^2$ *и, кроме того*

$$\overline{\lim_{n\to\infty}} \frac{1}{v(n)} \left| \frac{1}{P_n} \sum_{k=0}^{n} p_k \sum_{\nu=0}^{k} c_\nu \Phi_\nu(x) \right| = \infty$$

почти всюду в интервале $\langle 0, 1\rangle$.

Е. МИОДУШЕВСКИЙ. **О ДВУ-ОДНОЗНАЧНЫХ НЕПРЕРЫВНЫХ ФУНКЦИЯХ** . . . . . стр. 129—132

Приводится ряд результатов, относящихся к так называемым дву-однозначным функциям $f: X \to Y$, т.е. функциям, принимающим каждое свое значение точно два раза. Предполагается X локально компактным сепарабельным топологическим пространством, а Y — пространством типа L^* или гаусдорфовым пространством. Главная теорема сводит исследование дву-однозначных функций на так называемых псевдоокрестностях так называемых псевдо-эвклидовых точек к исследованию на них непрерывных инволюций. Один из полученных результатов указывает на существование дву-однозначных функций на эвклидовых пространствах E^n размерности $n \geqslant 2$.

Р. БИТТНЕР, **АЛГЕБРАИЧЕСКИЕ СВОЙСТВА ЛИНЕЙНЫХ УРАВНЕНИЙ С АБСТРАКТНОЙ ПРОИЗВОДНОЙ** . . . . . стр. 133—139

Под абстрактной производной понимается линейное отображение S пространства C^1 в C^0 ($C^1 \subset C^0$). Вводя понятие обобщенного решения линейного уравнения с производной $A_1 Sx + A_0 x = f$, где A_1, A_0 — эндоморфизмы C^0, $f \in C^0$, рассматриваем операции решающие $X_1 c_1 + \ldots + X_n c_n$, определенные на пространстве постоянных c_i, для которых $Sc_i = 0$, и принимающие значения из пространства решений однородного уравнения. Скажем, что решающие операции операторно независимы, если условие $X_1 c_1 + \ldots + X_n c_n = 0$ влечет за собой $c_1 = \ldots = c_n = 0$.

В работе приводятся теоремы, касающиеся количества и свойств операций, решающих операторно независимых для системы линейных уравнений уравнения n-того порядка, а также частных уравнений с абстрактными производными, при соответственно определенном порядке уравнения.

Р. БИТТНЕР, **АЛГЕБРАИЧЕСКИЕ СВОЙСТВА АБСТРАКТНЫХ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В АЛГЕБРАХ** . . стр. 141—142

Под абстрактной производной понимаем линейное отображение S алгебры C^1 в C^0 ($C^1 \subset C^0$) со свойствами

$$S(xy) = Sx \cdot y + x \cdot Sy \quad \text{для} \quad x, y \in C^1.$$

Как первичную операцию понимаем такую линейную операцую T, что $STx = x$ для $x \in C^0$. При $sx = x - TSx$ предполагается, кроме того, что

$$s(xy) = sx \cdot sy \quad \text{для} \quad x, y \in C^1.$$

Пусть $X_1 x_1, \ldots, X_n x_n$ где $Sx_1 = \ldots = Sx_n = 0$ будут операциями, принимающими в качестве значений решения системы n линейных урав-

нений с абстрактной производной или уравнений n-того порядка с абстрактной производной. Скажем, что эти операции операторно независимы, если

$$X_1 x_1 + \ldots + X_n x_n = 0 \supset x_1 = \ldots = x_n = 0 .$$

Работа содержит критерии необходимые и достаточные для операторной независимости решений вида $X_i x_i = a_i x_i$. А именно, соответствующий детерминант Вроньского, где вместо обыкновенной производной следует поместить абстрактную производную, не может быть делителем нуля.

Кроме того, в работе дается метод постоянной вариации для уравнения n-того порядка с числовыми коэффициентами.

Т. ГУСКОВСКИЙ, **О НОРМАЛИ К ГИПЕРПОВЕРХНОСТИ В АФИННОМ ПРОСТРАНСТВЕ** A_n . . . . . стр. 143—146

В работе строится афинная нормаль к гиперповерхности $\mathfrak{M}_{n-1}$ класса C^3 в афинном пространстве A_n. Индуцированная связность в смысле Леви-Чивита на этой гиперповерхности является эквиафинной.

В. РОТЕР, **О ГЕОДЕЗИЧЕСКОМ ОТОБРАЖЕНИИ РИМАНОВА ПРОСТРАНСТВА НА РЕКУРРЕНТНОЕ ПРОСТРАНСТВО** . . . . . стр. 147—149

Обобщая теорему Н. С. Синюкова [2], доказывается, что риманово пространство, которое можно отобразить геодезически на пространство с рекуррентной кривизной, имеет постоянную кривизну.

Т. ВАЖЕВСКИЙ, **УПРАВЛЯЕМЫЕ СИСТЕМЫ И КОНТИНГЕНСНЫЕ УРАВНЕНИЯ** . . . . . стр. 151—155

Автор вводит понятие контингенсного уравнения K „ассоциированного" с данной управляемой системой S и дает некоторые условия эквивалентности S и K.

Благодаря этому, некоторые известные результаты, полученные А. Маршом и С. К. Зарэмбой, касающиеся уравнения K можно применить к системе S. Этим путем получаются некоторые результаты относительно существования периферийного решения системы S, которое в минимальном времени встречает постоянную или подвижную цель.

Автор рассматривает нелинейный случай.

Соответствующие результаты автор применяет к случаю, когда S — линейно относительно переменных управления, но нелинейно относительно переменных, определяющих положение подвижной точки.

В. ОРЛИЧ, **НЕКОТОРЫЕ ЗАМЕЧАНИЯ ИЗ ТЕОРИИ МОДУЛЯРНЫХ ПРОСТРАНСТВ** I. . . . . стр. 157—162

S-выпуклым модуляром называем функционал $\varrho(x)$, определенный в линейном пространстве X и удовлетворяющий следующим условиям:

A. 1. $\varrho(0)=0$;

A. 2. $\varrho(\lambda x)=0$ влечет за собой $x=0$;

A. 3. $\varrho(-x)=\varrho(x)$;

A. 4. $\varrho(\alpha x+\beta y)\leqslant \alpha^s\varrho(x)+\beta^s\varrho(y)$ для $\alpha,\beta\geqslant 0$, $\alpha^s+\beta^s=1$.

Предметом работы является введение нормы в X при помощи S-выпуклого модуляра, а также некоторые свойства S-выпуклых модуляров и модуляров, определенных в предыдущих работах [2], [3].

А. ГЖЕГОРЧИК, А. МОСТОВСКИЙ и Ч. РЫЛЛЬ-НАРДЗЕВСКИЙ, **ОПРЕДЕЛЯЕМОСТЬ МНОЖЕСТВ В МОДЕЛЯХ АКСИОМАТИЧЕСКИХ ТЕОРИЙ** стр. 163—167

Используя топологический метод авторы доказывают теоремы Ганди, Крейзеля и Тэта [1], утверждающие, что только рекуррентные множества (гипер-арифметические) содержатся в каждой модели (ω-модели) теории множеств с рекуррентной аксиоматикой (типа π^1_1). Эти теоремы обобщаются на случай произвольных теорий первого ряда.

М. АЛЬТМАН, **ЛИНЕЙНЫЙ ИТЕРАЦИОННЫЙ МЕТОД С ВЕКТОРНЫМ ПАРАМЕТРОМ** . . . . . стр. 169—173

В работе предложен итерационный метод решения линейных уравнений в пространстве Гильберга. Характиристической чертой этого метода является то, что он содержит произвольный вектор в качестве параметра, от которого в свою очередь зависит скорость сходимости процесса. Вообще говоря, предложенный метод сходится быстрее, чем метод найскорейшего спуска.

М. АЛЬТМАН, **ИТЕРАЦИОННЫЙ МЕТОД ДЛЯ ЛИНЕЙНЫХ АЛГЕБРАИЧЕСКИХ УРАВНЕНИЙ** . . . . . стр. 175—177

Работа содержит некоторый итерационный метод приближенного решения систем линейных алгебраических уравнений. Сущность метода заключается в указании одного способа сведения решения системы к задаче о собственных векторах, к которой применяется известный метод последовательных итераций. Сверх того вводится еще произвольный вектор в качестве параметра.

А. ВИТКОВСКИЙ, **ЭЛЕКТРОННЫЕ СПЕКТРЫ ДИМЕРОВ.** I. **ХАРАКТЕР СЦЕПЛЕНИЯ** . . . . . стр. 179—183

Для простой модели димера (2, 2, 1-бициклогептадиен) произведено исследование характера сцепления электронных движений с вибрациями ядер. Продискутирован основной гамильтониан, описывающий вибронные состояния димера.

Оказалось, что несимметрические колебания ядер (колебания пар ядер в обоих мономерах с относительной разницей фаз) могут подвергаться сильному сцеплению с движением электронов, тогда как полносимметрические колебания ядер (согласующиеся в фазе колебания пар ядер в обоих мономерах) не сцепляются с электронным движением.

Продискутирован характер кривых потенциальной энергии для несимметрических колебаний ядер.

А. ВИТКОВСКИЙ. **ЭЛЕКТРОННЫЕ СПЕКТРЫ ДИМЕРОВ. II. ПЕРТУРБАЦИОННЫЕ РЕШЕНИЯ ОСНОВНОГО ВИБРОННОГО УРАВНЕНИЯ** стр. 185—187

Основное вибронное уравнение, продискутированное в первой работе из этой серии, решается для случаев сильного и слабого сцепления. Сильное сцепление имеет место тогда, когда резонансное воздействие сильно и вибронные уровни определяются главным образом путем электронных рассуждений. Слабое сцепление имеет место в таком случае, если электронное резонансное воздействие слабо и спектральные эффекты ограничиваются деталями вибрационной структуры полосатых спектров.

А. ВИТКОВСКИЙ. **ЭЛЕКТРОННЫЕ СПЕКТРЫ ДИМЕРОВ. III. ИНТЕНСИВНОСТИ** . . . . . стр. 189—191

Для спектра вычислены: полная дипольная интенсивность, положение центра тяжести и ширина спектра. Получены также выражения для очередных моментов спектра. Произведено сопоставление с соответствующими величинами, касающимися спектра мономеров.

Оказалось, что главным фактором, вызывающим изменение характеристики спектра димера, в сравнении с мономером, является параметр электронного резонансного сцепления.

Ю. МЫЦЕЛЬСКИЙ. **ОБ ОДНОМ МЕХАНИЗМЕ ПРИМЕСНОЙ ПРОВОДИМОСТИ. I. ЭНЕРГИЯ АКТИВАЦИИ ПРОВОДИМОСТИ** . . . стр. 193—197

Автор продискутировал несколько возможных интерпретаций так называемой аномалии ϵ_2, найденной в проводимости и в постоянной Галла в германии типа n и p при гелиевых температурах. Констатировано, что они не согласуются с результатами экспериментов.

Автор предложил новый механизм примесной проводимости, состоящий в скачках носителей с нейтральных центров примесей на ионизированные над валом кулоновского потенциала. Определена энергия активации проводимости и ее зависимость от концентрации примесей. Количественная согласованность с результатами экспериментов удовлетворительная.

Ю. МЫЦЕЛЬСКИЙ, **СБ ОДНОМ МЕХАНИЗМЕ ПРИМЕСНОЙ ПРОВОДИМОСТИ. II. ПРОВОДИМОСТЬ В СЛУЧАЕ СИЛЬНОГО ВОЗДЕЙСТВИЯ НОСИТЕЛЕЙ С ФОНОНАМИ** . . . . . стр. 199—205

Примесная проводимость в предложенном в первой части работы механизме ограничена продуктивностью двух процессов: возбуждением и диффузией носителей из нейтральных центров в ионизированные. Исследован второй процесс, который определяет проводимость в случаях сильного воздействия носителей с фононами. Получена, во первых, частота скачков носителей с нейтральных центров на ионизированные и, во вторых, электрическая проводимость. Сравнение с экспериментальными данными для германия типа n и p проявляет качественную согласованность. Для количественного сравнения не имеется до настоящего времени соответствующих экспериментальных данных.

И. ВЕРЛЕ, **О НЕКОТОРЫХ ПРОБЛЕМАХ ФОРМАЛИЗМА *S*-МАТРИЦЫ ДЛЯ НЕУСТОЙЧИВЫХ ЧАСТИЦ** . . . . . стр. 207—210

В работе показано, что некоторые трудности, присущие введению в S-матрицу членов, описывающих распад неустойчивых частиц, могут быть устранены путем введения при использовании ковариантного метода размытия энергии неустойчивой частицы.

Установлены автором следующие из унитарности S-матрицы связи между распадом частицы и рассеянием конечных продуктов распада. Показано, чем необходимо пренебречь, чтобы получить известную связь между относительной фазой параметров и фазовыми сдвигами рассеивания.

ЖИРО КОБА, **ВЕСОВОЙ ФАКТОР МОМЕНТА ИМПУЛЬСА В СТАТИСТИЧЕСКОЙ ТЕОРИИ МНОЖЕСТВЕННОГО ОБРАЗОВАНИЯ ЧАСТИЦ**. I. стр. 211—216

В работе даются общие свойства, методы вычисления и рекуррентная формула для весового фактора момента импульса $Z_n(l_1, l_2, \ldots, l_n)$, играющего важную роль в статистической теории множественного образования частиц с учетом сохранения момента импульса, который определяется из соотношения

$$D^{(l_1)} \times D^{(l_2)} \times \ldots \times D^{(l_{n-1})} = \sum_{\lambda} Z_n(l_1, l_2, \ldots, l_{n-1}, \lambda) \cdot D^{(\lambda)}.$$

Р. ЖЕЛЯЗНЫ, и А. КУШЕЛЛЬ, **ПРОБЛЕМА МИЛЬНА ДЛЯ ДВУХ СОПРИКАСАЮЩИХСЯ ПОЛУПРОСТРАНСТВ** . . . . . стр. 217—220

В работе представлено новое применение метода Кейса, развернутого в односкоростной теории транспорта нейтронов, по отношению к проблеме Мильна для двух соприкасающихся полупространств. Среды обоих полупространств характеризуются различной длиной свободного пути и чистом вторичных нейтронов по столкновению, т.е. разными параметрами l и c.

В. ЗЕНТЭК, **КВАНТОВО-ТЕОРЕТИЧЕСКИЙ МЕТОД ПОЛУЧЕНИЯ МАГНИТНОЙ СТРУКТУРЫ АНИЗОТРОПНЫХ ФЕРРОМАГНИТНЫХ МОНОКРИСТАЛЛОВ** . . . . . стр. 221—224

В работе приводится однородный квантово-механический метод минимализации для исчисления доменных структур анизотропических ферромагнитных монокристаллов, обоснованный на известном вариационном методе приближения. Уже самое простое приближение ведет к лучшим, чем известные до сих пор полученные результаты, вычисленные при применении полуклассических методов (см. напр. [2], [3]).

В качестве примера приводятся результаты, полученные для кристаллографического направления [100] в простой кубической решетке.

Я. МОЗЖИМАС и Я. ЖЕВУСКИЙ, **ОБ ИНВАРИАНТНОМ ДИФФЕРЕНЦИАЛЬНОМ УРАВНЕНИИ ВТОРОГО ПОРЯДКА В СПИНОРНОМ ПРОСТРАНСТВЕ** стр. 225—228

В работе показано, что общее дифференциальное уравнение второго порядка в спинорном пространстве, инвариантное по отношению к группе $c \cdot c' \cdot a$ этого пространства, является аналитическим продолжением специального дифференциального уравнения, рассмотренного подробно в [1].

Б. ГИЛЬЧЕР, **ВЛИЯНИЕ p-ИЗУЧЕНИЯ ^{60}Co НА ДИЭЛЕКТРИЧЕСКИЕ СВОЙСТВА ТРИГЛИЦИНСУЛЬФАТА** . . . . . стр. 229—234

Исследовано влияние γ-излучения ^{60}Co на диэлектрическую проницаемость, электрическую поляризацию и изменение температуры точки Кюри для триглицинсульфата. Диэлектрическая проницаемость облученных образцов в точке Кюри понижается (рис. 1); температура точки Кюри смещается в сторону более низких температур (рис. 2а). Петли гистерезиса облученного триглицинсульфата меняют свою форму, начиная с нормальной петли, через двойную, до линейной зависимости электрической поляризации от напряженности электрического поля (рис. 3). Облучение γ-лучами ^{60}Co происходило в точке, в которой мощность дозы была порядка $10^3\ r/h^{-1}$.

Г. ЛОЖИКОВСКИЙ и Г. МЕНЧИНСКАЯ, **ПОПЫТКА ПОЛУЧЕНИЯ ЭЛЕКТРОЛЮМИНЕСЦСНЦИИ ОРГАНОФОСФОРОВ** . . . . . стр. 235—236

Авторами была произведена попытка получения электролюминесценции органических красителей в вакууме 10^{-5} *мм* Hg.

Оказалось, что свечение ячеек содержащих краситель, адсорбированный на целлофане, неокрашенный целлофан, а даже только слюду, было почти одинаковое.

На основании этих опытов авторы пришли к заключению, что наблюдаемое свечение не происходит от органических красителей.

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